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Mixed Discontinuous Galerkin Methods for Elasticity

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ABSTRACT

In this talk, we review existing discontinuous Galerkin(DG) methods for elasticity and introduce a new formulation based on mixed finite elements which preserves the important property of being locally conservative. We then highlight the subtle and important differences between Interior Penalty(IP) and mixed FEM based methods. With regards to mixed method for elasticity, choices of approximating spaces for primal and dual variable is non-trivial for two main reasons, namely the compatibility between the two spaces(inf-sup condition) and a symmetry condition for interpolating stress tensor in such a space. We will discuss and address these issues and show that the resulting mixed scheme is highly simplified, extremely robust and optimally convergent in global norms.

Interior Penalty (IP) based DG methods for linear elasticity and quasi-static viscoelasticity have been proposed by Riviére in [1] based on early work by and Baumann, Oden and Babuska [2].Symmetric and non-symmetric versions of IPG method have been further studied by Wihler in [3] and others and are shown to be locking free(robust) for incompressible materials. The method is however based on using non symmetric flux terms(on internal boundaries) and interior penalty for imposing weak inter-element continuity and these terms render it locally non-conservative. In more recent work by Tenyck and Lew [4] IP based DG method was used for nonlinear elasticity problems.

We define a Lipschitz continuous domain Ω, displacement field u_i, gradient of the displacement field ϵ_{ij}, strains as ϵ^s_{ij}, Lame's parameters(λ, μ) and stresses σ_{ij}. Now using the finite element spaces:

$$V^h = \{v_i \in L^2(\Omega) : v_i|_{\Omega_k} \in \mathcal{P}^m(\Omega_k), \Omega_k \in \mathcal{T}_h, \forall\, i \in \{1,2\}\}$$
$$W^h = \{s_{ij} \in L^2(\Omega) : s_{ij}|_{\Omega_k} \in \mathcal{P}^n(\Omega_k), \Omega_k \in \mathcal{T}_h, \forall\, i,j \in \{1,2\}\}$$

where, $\mathcal{P}^m(\Omega_k)$ denotes the standard order 'm' polynomial space based on hierarchical(Legendre) basis functions, we define mixed DGFEM forms. Find $(u_i, \epsilon_{ij}) \in V^h \times S^h$ such that $\forall (v_i, s_{ij}) \in V^h \times S^h$, satisfying either (1) and (2) or (3) and (4)

$$a(\epsilon, S) + b(u, S) - < u, S >= 0 \qquad (1)$$
$$c(\sigma, v) - < \sigma_{ij} n_j, u_i >= f(v) \qquad (2)$$
$$a(\epsilon, S) + B(u, S) = 0 \qquad (3)$$
$$c(\sigma, v) - < \sigma_{ij} n_j, u_i > + \Gamma(u, v) = f(v) \qquad (4)$$

where

$$a(\epsilon, S) = \int_{\Omega_k} \epsilon_{ij} S_{ij} d\Omega \;\; b(u, v) = \int_{\Omega_k} u_i S_{ij,j} d\Omega \;\; B(u, v) = \int_{\Omega_k} u_{i,j} S_{ij} d\Omega$$
$$c(\sigma, v) = \int_{\Omega_k} \sigma_{ij}(u) \epsilon_{ij}(v) d\Omega \;\; \Gamma(u, v) = \frac{c\mu}{h} \int_{\partial\Omega_k} [\![u_i]\!][\![v_i]\!] ds \;\; < u, v >= \oint_{\partial\Omega_k} uvds$$

For these formulations we prove consistency and convergence in appropriate global norms and local conservations. We also illustrate the convergence behavior using a set of numerical examples. The results show good convergence. We also investigate numerical behavior of the *inf-sup* constants and condition numbers.

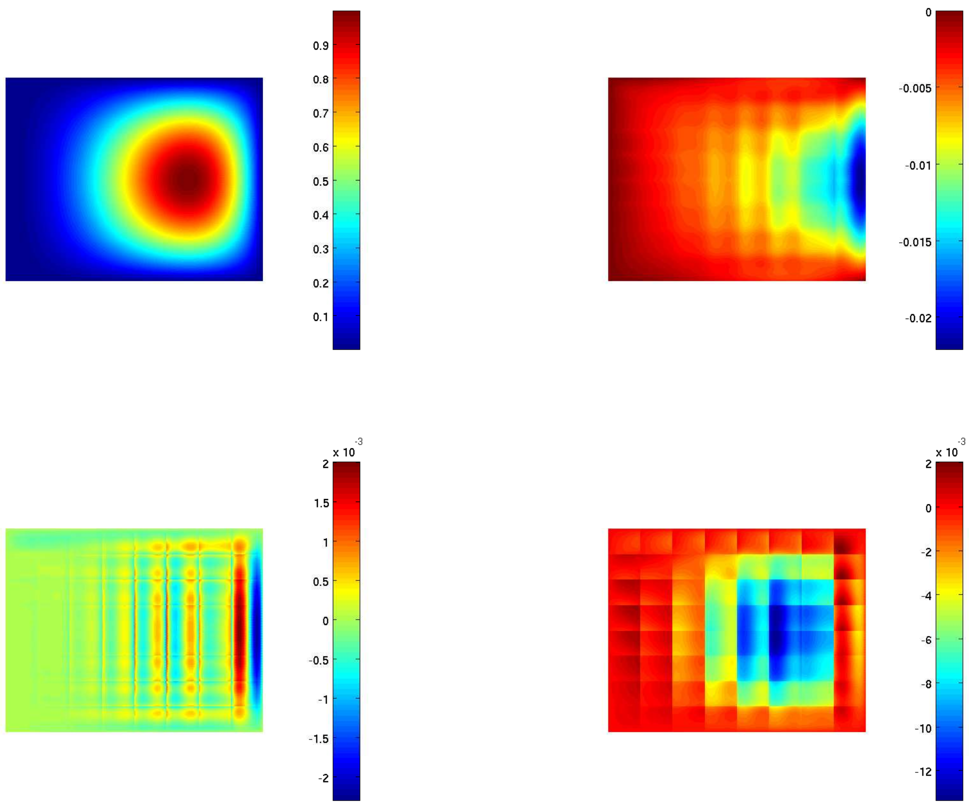

Figure 1: Comparison of error in various DGFEM solutions with $m = 2$ on 8x8 mesh

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