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COMPUTATIONAL METHODS FOR SPDE CONTROL PROBLEMS

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ABSTRACT

In order to provide a concrete setting for our discussion, we consider the problem

$$\begin{aligned} -\nabla\cdot\big(\kappa(\omega,x)\nabla u(\omega,x)\big)-c(\omega,x)u(\omega,x) &= f(\omega,x) \quad \text{in } \Omega\times D \\ u(\omega,x) &= 0 \qquad\quad\; \text{on } \Omega\times\partial D, \end{aligned} \tag{1}$$

where we have the correlated random fields $\kappa(\omega,x)$, $c(\omega,x)$, and $f(\omega,x)$ with $\omega\in\Omega$ denoting an elementary event in a probability space Ω and $x\in D$ denoting a point in the spatial domain D. Then, solutions $u(\omega,x)$ of the elliptic SPDE (1) are also random fields. The case $\kappa>0$ and $c=0$ almost surely (a.s.) models random diffusion problems such as flows in porous media or thermal problems in media with a random thermal conductivity. The case $c>0$ and $\kappa=1$ a.s. models acoustic waves in random media. The case $\kappa>0$ and $c\neq 0$ a.s. models (linear) diffusion-reaction problems with a random diffusion coefficient and a random "reaction rate." In all cases, $f(\omega,x)$ is a random forcing field that, e.g., models heat sources and sinks in a random thermal diffusion problem.

First, we consider an *optimal control* problem for involving (1). Again, for the sake of concreteness, we define the simple cost or objective functional

$$\mathbb{E}\left[\frac{1}{2}\|u(\omega,\cdot)-\overline{u}(\omega,\cdot)\|^2_{L^2(D)}+\frac{\alpha}{2}\|f(\omega,\cdot)\|^2_{L^2(D)}\right], \tag{2}$$

where $\mathbb{E}(\cdot)$ denotes the expected value and $\overline{u}$ is a given target random field. In some applications, $\overline{u}$ may be deterministic. By controlling the solution in this natural norm we also control the statistics of the solution, e.g. the expected value of the solution, given by

$$\|\mathbb{E}[u-\overline{u}]\|^2_{L^2(D)}\leq\mathbb{E}\left[\|u-\overline{u}\|^2_{L^2(D)}\right]\leq\|u-\overline{u}\|^2_{L^2_P(\Omega;L^2(D))}\,.$$

The optimal control problem is then given as follows: given the random fields $\kappa(\omega,x)$ and $c(\omega,x)$, minimize the functional (2) over all $f\in L^2_P(\Omega,L^2(D))$ subject to $u\in L^2_P(\Omega;H^1_0(D))$, satisfying a weak formulation of (1).

The second problem we consider is a *parameter identification problem.* We now suppose that the random field f in (1) is given but that either or both random fields κ or c are not known. To be concrete, let

us assume that it is κ. Suppose also that there is a set of measurements $\{\eta(\omega, x) \ : \ \omega \in \Omega, x \in D\}$, or a discrete set of such data at a finite number of points in D, or even on a small subset D° of the physical domain D. The measurements correspond to some quantity of interest $Q(u)$ that depends on the solution u of the SPDE (1); this quantity of interest may not depend on x if, e.g., it involves an integral over the spatial domain D, but for our discussion, we will keep to the more general case where it does, e.g., the quantity of interest is u itself. We wish to use the given measured data to help us determine some information about the coefficient random field κ. To this end, we define the functional

$$\mathbb{E}\left[\left\|Q\big(u(\omega,\cdot)\big) - \eta(\omega,\cdot)\right\|^2\right], \tag{3}$$

where the norm used is appropriate for the quantity of interest $Q(\cdot)$. Given the random fields f and c, we then pose the problem of finding an optimal coefficient random field κ and an optimal state random field u such that the functional (3) is minimized, subject to u and κ satisfying a weak form of the state system (1). An simpler version of this problem would involve the identification of random parameters appearing in the coefficient of the SPDE instead of random fields.

We develop, analyze, and implement stochastic collocation finite element methods (SCFEMs), using adaptive sparse grid sampling, with and without reduced-order modeling approaches in physical space, for the approximation of solutions of the optimality systems arising from the above stochastic optimal control and parameter identification problems. For example, for the optimal control problem we show that the optimality system is given by the state system (1), the adjoint system

$$\begin{aligned} -\nabla \cdot \big(\kappa(\omega, x)\nabla\xi(\omega, x)\big) - c(\omega, x)\xi(\omega, x) &= \widehat{u}(\omega, x) - \overline{u}(\omega, x) && \text{in } \Omega \times D \\ \xi(\omega, x) &= 0 && \text{in } \Omega \times \partial D, \end{aligned}$$

where $\xi \in L^2_P(\Omega; H^1_0(D))$ is the adjoint or co-state variable, and the optimality condition

$$\widehat{f}(\omega, x) = -\frac{1}{\alpha}\,\xi(\omega, x) \qquad \text{a.e. in } \Omega \times D.$$

Combining the SCFEMs with standard (deterministic) optimal control methods allows for decoupled (physical and probabilistic) computations of the stochastic optimality system, where, at each collocation point, deterministic techniques and analyses can be utilized. The *advantage* of our approach over classical methods is that, considering random input data, we control statistical moments (mean value, variance, covariance, etc.) or even the whole probability distribution of physical quantities of interest.

It is important to emphasize that our ideas can easily be applied to solve more complicated nonlinear and/or time-dependent optimal control problems involving SPDE constraints and to the case where the control random field is replaced by random parameter control variables; in fact, everything simplifies in that case. Note that, in general, we cannot use truncated Karhunen-Loevy expansions to reduce the case of controls f that are random fields to a random parameter problem since we do not know the covariance structure of the unknown control. In fact, the determination of that covariance structure is one of outputs of our optimal control strategy. However, one can greatly reduce the complexity of the problem if one is only interested in determining some lower-order statistics, e.g., mean values, variances, and other low-order moments; in this case, one can reduce the number of degrees of freedom needed to describe the random control field f by approximating it by a truncated moment expansion.