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An *a posteriori* error estimator for a quadratic C^0 interior penalty method for the biharmonic problem

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ABSTRACT

C^0 interior penalty methods for fourth order problems were introduced by Engel *et al.* in [1] and further studied in [2,3,4,5]. These methods use standard C^0 finite elements for second order problems and they preserve the symmetric positive-definiteness of the continuous problems. Moreover, the derivations of these methods are very straightforward, using only integration by parts, symmetrization and penalization. Therefore C^0 interior penalty methods have various advantages over the classical approaches of conforming finite elements, mixed finite elements and nonconforming finite elements.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $f \in L_2(\Omega)$. Consider the model biharmonic problem of finding $u \in H_0^2(\Omega)$ such that

$$\sum_{i,j=1}^{2} \int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}\, dx = \int_\Omega f v\, dx \qquad \forall\, v \in H_0^2(\Omega). \tag{1}$$

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω and $V_h \subset H_0^1(\Omega)$ be the Lagrange P_2 finite element space associated with $\mathcal{T}_h$. The quadratic C^0 interior penalty method for (1) is to find $u_h \in V_h$ such that

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h} \sum_{i,j=1}^{2} \int_T \frac{\partial^2 u_h}{\partial x_i \partial x_j} \frac{\partial^2 v_h}{\partial x_i \partial x_j}\, dx + \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 u_h}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] ds \\ &\quad + \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right] ds + \sum_{e\in\mathcal{E}_h} \frac{\eta}{h_e} \int_e \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right] \left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] ds = \int_\Omega f v_h\, dx \end{aligned} \tag{2}$$

for all $v_h \in V_h$, where $\mathcal{E}_h$ (respectively $\mathcal{E}_h^i$) is the set of the edges (respectively interior edges) of $\mathcal{T}_h$,

$$\left[\!\!\left[\frac{\partial v_h}{\partial n} \right]\!\!\right] \quad \text{and} \quad \left\{\!\!\left\{ \frac{\partial^2 v_h}{\partial n^2} \right\}\!\!\right\}$$

denote the jump of the normal derivative and the mean of the second order normal derivative across the edges respectively, h_e is the length of the edge e, and $\eta \geq 1$ is a sufficiently large penalty parameter.

In this talk we discuss a residual-based *a posteriori* error estimator

$$\mathbb{E}(u_h) = \Big(\sum_{T \in \mathcal{T}_h} h_T^4 \|f\|^2_{L^2(T)} + \sum_{e \in \mathcal{E}_h} \frac{\eta^2}{h_e} \int_e \left[\!\!\left[\frac{\partial u_h}{\partial n} \right]\!\!\right]^2 ds + \sum_{e \in \mathcal{E}_h^i} h_e \int_e \left[\!\!\left[\frac{\partial^2 u_h}{\partial n^2} \right]\!\!\right]^2 ds \Big)^{1/2},$$

for the solution u_h of (2), where h_T denotes the diameter of T. This error estimator is both reliable and efficient with respect to the norm $\| \cdot \|_{H^2(\Omega,\mathcal{T}_h)}$ defined by

$$\|v\|^2_{H^2(\Omega,\mathcal{T}_h)} = \sum_{T \in \mathcal{T}_h} \sum_{i,j=1}^{2} \Big\| \frac{\partial^2 v}{\partial x_i \partial x_j} \Big\|^2_{L_2(T)} + \sum_{e \in \mathcal{E}_h} \frac{\eta}{h_e} \Big\| \left[\!\!\left[\frac{\partial v}{\partial n} \right]\!\!\right]^2 \Big\|^2_{L_2(e)}.$$

More precisely, we have

$$\|u - u_h\|_{H^2(\Omega,\mathcal{T}_h)} \leq C_1 \, \mathbb{E}(u_h), \tag{3}$$

$$\mathbb{E}(u_h) \leq C_2 \eta^{1/2} \Big(\|u - u_h\|^2_{H^2(\Omega,\mathcal{T}_h)} + \sum_{T \in \mathcal{T}_h} h_T^4 \|f - \bar{f}\|^2_{L_2(T)} \Big)^{1/2}, \tag{4}$$

where $\bar{f}$ is the L_2-orthogonal projection of f onto the space of piecewise constant functions with respect to $\mathcal{T}_h$, and the constants C_1 and C_2 depend only on the shape regularity of $\mathcal{T}_h$.

We will outline the derivations of (3) and (4), and present numerical results that demonstrate the performance of the error estimator. Details can be found in [6].

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