High order discontinuous Galerkin schemes on general 2D manifolds

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Let us consider the following simplified problem. We consider the unique parabola that passes through the three points $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$. One possible representation of the parabola can be written as

$$\vec{x}(\xi) = \underbrace{\frac{(\xi-1)\xi}{2}}_{L_1^2(\xi)}\ \vec{x}_1 + \underbrace{(1-\xi^2)}_{L_2^2(\xi)}\ \vec{x}_2 + \underbrace{\frac{(\xi+1)\xi}{2}}_{L_3^2(\xi)}\ \vec{x}_3.$$

It maps the interval $\xi \in [-1, 1]$ onto the parabola using the so called quadratic Lagrange polynomials.

The tangent vector

$$\vec{t}(\xi) = \frac{\partial \vec{x}}{\partial \xi} = \frac{(2\xi-1)}{2}\ \vec{x}_1 - 2\xi\ \vec{x}_2 + \frac{(2\xi+1)}{2}\ \vec{x}_3.$$

can be written in a more convenient form

$$\vec{t}(\xi) = \underbrace{\frac{(1-\xi)}{2}}_{L_2^1(\xi)}\ \underbrace{\left(-\frac{3}{2}\vec{x}_1 + 2\vec{x}_2 - \frac{1}{2}\vec{x}_3\right)}_{\vec{t}_1} + \underbrace{\frac{(1+\xi)}{2}}_{L_2^1(\xi)}\ \underbrace{\left(\frac{1}{2}\vec{x}_1 + -2\vec{x}_2 + \frac{3}{2}\vec{x}_3\right)}_{\vec{t}_2}$$

than involves the two tangent vectors $\vec{t}_1 = \vec{t}(-1)$ and $\vec{t}_2 = \vec{t}(1)$ at the end points of the parabola that are interpolated using Lagrange shape functions of one order lower than the parametrization, i.e. of order one.

Now, our aim is to build an vector-valued interpolation $\vec{v}(\xi)$ that is always tangent to the parabola. Let us consider the following polynomial expansion at order p

$$f(\xi) = \sum_{i=1}^{p+1} L_i^p(\xi) f_i$$

Multiplying $\vec{t}$ by f allows to build a parametrizes tangent vector to the parabola:

$$\vec{v}(\xi) = \vec{t}(\xi) f(\xi) = \sum_{i=1}^{p+1} L_i^p(\xi) f_i \left(L_1^1(\xi)\vec{t}_1 + L_2^1(\xi)\vec{t}_2 \right).$$

For sake of simplicity, let us expand this last expression for $p = 1$:

$$\vec{v}(\xi) = f_1 \underbrace{\left[L_1^1(\xi)^2 \vec{t}_1 + L_1^1(\xi) L_2^1(\xi) \vec{t}_2 \right]}_{\vec{T}_1(\xi)} + f_2 \underbrace{\left[L_1^1(\xi) L_2^1(\xi) \vec{t}_1 + L_2^1(\xi)^2 \vec{t}_2 \right]}_{\vec{T}_2(\xi)}. \quad (1)$$

Equation (1) is a vector valued finite element approximation that make use of quadratic vector valued shape functions $\vec{T}_1(\xi)$ and $\vec{T}_2(\xi)$ and of scalar coefficients f_1 and f_2. Clearly, we have $\vec{T}_1(-1) = \vec{t}_1$ and $\vec{T}_2(1) = \vec{t}_2$ so that our approximation looks very much like a classical finite element approximation: f_1 and f_2 can be interpreted as vector amplitudes at element end points. Disappointingly, this way of approximating fields cannot be used in a finite element framework because it does not fullfill the simpliest patch test: it is easy to see that (1) does not allow to interpolate a vector of constant amplitude on the parabola.

It is possible to remedy to that problem using unit tangent vectors:

$$\vec{v}(\xi) = \frac{\vec{t}(\xi)}{\|\vec{t}(\xi)\|} f(\xi) = \sum_{i=1}^{p+1} f_i \frac{L_i^p(\xi)(L_1^1(\xi)\vec{t}_1 + L_2^1(\xi)\vec{t}_2)}{\sqrt{L_1^1(\xi)^2 \left\|\vec{t}_1\right\|^2 + 2L_1^1(\xi)L_2^1(\xi)\vec{t}_1 \cdot \vec{t}_2 + L_2^1(\xi)^2 \left\|\vec{t}_2\right\|^2}}.$$

This approximation allows to make the amplitude of $\vec{v}$ to vary polynomially while remaining tangent to the parabola. Of course, vector shape functions are not polynomials, making finite element assembly process more expensive.

In this paper, we start by extending this way of interpolating vectors to 2D manifolds. We apply this technique to Discontinuous Galerkin discretization of the rotating shallow waters equations. This approach has the advantage to avoid the use of Lagrange multipliers: here, vectors are naturally tangent to the manifold. Then, we show how to solve problems on curvilinear meshes for which the normal may not be continuous through element interfaces. The classical Williamson test cases will serve as examples to assert the method.