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INTEGRATION SCHEME FOR WAVE BOUNDARY ELEMENTS

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ABSTRACT

We address problems of three-dimensional wave scattering governed by the Helmholtz equation

$$\nabla^2\phi(x) + k^2\phi(x) = 0, \qquad x \equiv (x, y, z) \in \Omega \tag{1}$$

in which ϕ is a complex valued potential field that we seek in Ω. The domain is bounded internally by a scatterer of boundary Γ, which is impinged by an incident plane wave ϕ^I. We denote with k the wave number. The solution is sought subject to boundary conditions of the general form

$$\nabla\phi(x) \cdot n = \nu\phi(x) + \beta, \quad x \in \Gamma \tag{2}$$

in which n is the unit normal directed outward from Ω. Classical approaches lead to the expression of the differential equation in boundary integral form as the sum of layer potentials

$$\frac{1}{2}\phi(x_0) + (\mathcal{K}\phi)(x_0) - \nu(\mathcal{S}\phi)(x_0) = \phi^I(x_0) + (\mathcal{S}\beta)(x_0), \quad x_0 \in \Gamma \tag{3}$$

where $(\mathcal{S}\phi)(x_0)$ and $(\mathcal{K}\phi)(x_0)$ are the usual single and double layer potential operators. Wave boundary elements use Partition of Unity [1] ideas, expressing ϕ at node j as a linear combination of plane waves $A_{jm}e^{ikd_{jm}\cdot x}, m = 1, ..., M$. This has been found to provide marked benefits in accuracy and efficiency [2], but suffers from a requirement to evaluate highly oscillatory boundary integrals of the form

$$I_{jm}(x_0) = \int_{-1}^{+1}\int_{-1}^{+1} f(x_0, \xi, \eta)e^{ikg_{jm}(x_0,\xi,\eta)}d\xi d\eta \tag{4}$$

Here, $f(x_0, \xi, \eta)$ represents the slowly varying terms (deriving from the non-oscillatory part of the Green's function, the Jacobian and the shape function) and

$$g_{jm}(x_0, \xi, \eta) := r + d_{jm} \cdot x \tag{5}$$

is the direction along which oscillation takes place, $r := |x - x_0|$ being the usual radial coordinate in boundary integral methods. We present an integration scheme based on a numerical coordinate transformation, in which the element is partitioned into quadratic triangular and quadrilateral subdivisions

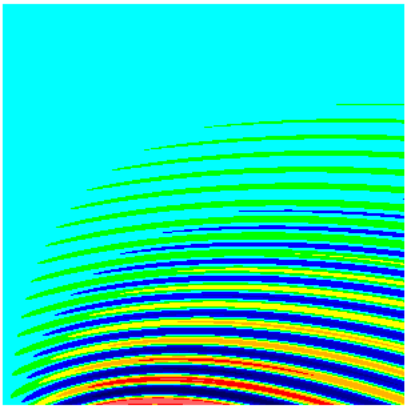

Figure 1: Contours of Re(integrand)

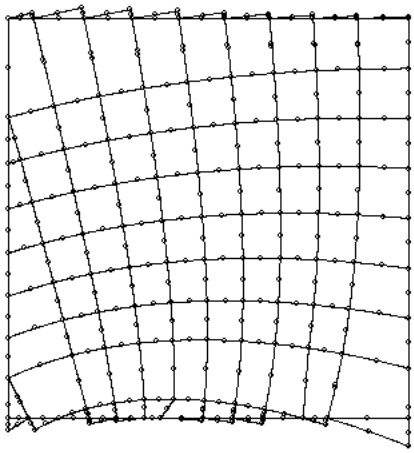

Figure 2: Element subdivisions

k	High order		New scheme		New scheme (nsd)	
	NGP	ε	NGP	ε	NGP	ε
250	6 412 332	2.92 x 10^{-12}	123 104	4.47 x 10^{-12}	18 369	6.46 x 10^{-13}
500	25 649 328	1.23 x 10^{-11}	246 256	1.64 x 10^{-11}	19 909	7.61 x 10^{-12}
1000	102 597 312	1.37 x 10^{-12}	492 492	1.66 x 10^{-11}	22 987	7.62 x 10^{-12}

Table 1: Number of Gauss points (NGP) and relative error ε for flat triangular element

bounded by contours of $g_{jm}(x_0, \xi, \eta) = const.$ and by a set of lines orthogonal to these contours, as illustrated in Figures 1 and 2. Each subdivision may be treated using a high order quadrature in the oscillatory direction but a low order quadrature in the direction along the contours of $g_{jm}(x_0, \xi, \eta) = const.$ Typically only 8 Gauss points are required per subdivision in the non-oscillatory direction (seemingly independent of wavelength) to give results of similar accuracy to high order integration of 30 Gauss points per wavelength but using far fewer Gauss points (Table 1 shows illustrative results for a triangular element having side lengths 2, $\sqrt{2}$, $\sqrt{6}$). Some subdivisions around the periphery are seen in Figure 2 to extend outside the parent element; the addition and subtraction of the contributions from these extensions removes difficulties found using the low order scheme in these subdivisions. The new scheme exhibits complexity $\mathcal{O}(k)$, comparing with $\mathcal{O}(k^2)$ for the high order scheme.

The complexity can be improved to little more than $\mathcal{O}(1)$ by making use of the method of numerical steepest descent (nsd) [3] in the oscillatory direction, but this becomes a cost-effective enhancement in CPU time only for integrals over boundary elements spanning more than 200 to 300λ.

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