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RIVER POLLUTION REMEDIATION: AN OPTIMAL CONTROL PROBLEM

Lino J. Alvarez-Vázquez[1], * Aurea Martínez[1], Miguel E. Vázquez-Méndez[2] and Miguel A. Vilar[2]

[1] Universidad de Vigo
ETSI Telecomunicación
36310 Vigo. Spain
{lino,aurea}@dma.uvigo.es

[2] Universidad de Santiago de Compostela
Escola Politécnica Superior
27002 Lugo. Spain
{ernesto,miguel}@usc.es

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ABSTRACT

In the last times, very strict legislative requirements concerning to the wastewater disposal in rivers have been established by the governments, so that all wastewater discharged into a river must first be treated in a purifying plant, in order to reduce its pollutants levels. Despite all these legal regulations, many of our rivers keep levels of pollutants higher than the allowed thresholds, since the river basins are not capable of assimilating all the wastewater disposed there. The most common practice for the remediation of these polluted river sections consists of the injection of a given amount of clear water from a reservoir in a nearby point. In this process of increasing the river flow by controlled releases from reservoirs, the main problem consists (once the injection point p is chosen by geophysical reasons) of finding the minimum quantity of water Q which is needed to be injected into the river section in order to purify it up to a desired level.

In this talk we make use of the powerful tools of the mathematical modelling and the optimal control of PDE in order to determine this minimal quantity of injected water such that we can be able to ensure that the contaminant concentration in the river is lower than a fixed threshold. (The authors have previously analyzed and solved a related problem - the water conveyance problem [1] - for the case of a boundary control problem posed on a two-dimensional area of shallow water). We formulate our ecological problem as a hyperbolic optimal control problem with control constraints: Let us take a river L meters in length, and consider E tributaries flowing into the river (located at points $e_1, \ldots, e_E$), V wastewater discharges (located at points $v_1, \ldots, v_V$) coming from purifying plants, and a point p where clear water is discharged from a nearby reservoir.

Having in mind that we want to control pollution in the river section corresponding to $[p, L]$ for a time interval of length T, and since we are going to consider only one-dimensional changes along the direction of flow in the river, for each $(x,t) \in [0,L] \times [0,T]$ let us denote by $A(x,t)$ the area of the section occupied by water (*wet section*) that is assumed to remain positive for any time; let us denote by $u(x,t)$ the average velocity in the wet section x meters from the source and t seconds from the moment the control is initiated; let us denote by $q(x,t)$ the flow rate across the section (that is, $q(x,t) = A(x,t)u(x,t)$); and let us denote by $c(x,t)$ the quantity of a generic pollutant in the wet section (that is, $c(x,t) = A(x,t)\rho(x,t)$, with $\rho(x,t)$ the averaged pollutant concentration). The evolution of the wet area $A(x,t)$, the flow rate $q(x,t)$, and the quantity of pollutant $c(x,t)$ is given by the 1D shallow water equations coupled with the pollutant concentration equation [2].

Our control will be $Q(t)$, the flow rate of clear water injected in p all along the time interval $[0,T]$. By technological reasons we consider only the positive fluxes in the set of admissible controls given by $U_{ad} = \{Q \in L^2(0,T) \,:\, 0 \leq Q \leq Q_{max}\}$, since we are just injecting (not extracting) clear water, and the quantity of injected water must be bounded.

Then, we formulate the control problem considering as the cost functional the total amount of clear water injected through the point p together with a measure in the region of the river starting from point p of the contaminant concentration which remains higher than a given fixed threshold c_{max}. Thus, we define the cost function:

$$J(Q) = \frac{\varepsilon}{2}\int_0^T Q(t)^2\,dt + \frac{\mu}{2}\int_0^T\int_p^L (c(x,t) - c_{max})_+^2\,dx\,dt.$$

So the problem, denoted by $(\mathcal{P})$, of the optimal water injection for the purification of a polluted section in a river consists of finding the control flux $Q \in U_{ad}$ of injected clear water in such a way that, verifying the state system, minimizes the cost function J.

By means of the introduction of the adjoint state (w, P, s), we can obtain a formal first-order optimality condition characterizing the solutions of the problem $(\mathcal{P})$. Moreover, this relation will also allow us to compute the gradient of the cost function $J'(Q)$, which can be used in order to solve numerically the control problem $(\mathcal{P})$ by minimizing the cost function *via* any gradient-type algorithm, for instance, an interior-point method.

In order to compute the value of the cost function J and its gradient J' at any control Q, we need to obtain the numerical solution of the state and of the adjoint systems. The discrete approximations of the state variables (A, q, c) (and in a similar way of the adjoint variables (w, P, s)) can be obtained with a finite elements/finite differences discretization whose complete mathematical details are introduced. Finally, an alternative optimization algorithm is also proposed (the gradient-free Nelder-Mead algorithm), and numerical results are presented for a realistic problem.

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