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ASYMPTOTIC HOMOGENIZATION OF HEMIVARIATIONAL INEQUALITIES IN ELASTICITY

Stanisław Migórski[1] and * Anna Ochal[1]

[1] Jagiellonian University
Faculty of Mathematics and Computer Science, Institute of Computer Science
ul. Nawojki 11, 30072 Krakow, Poland
email: {migorski, ochal}@softlab.ii.uj.edu.pl

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ABSTRACT

In this paper we study the homogenization of boundary hemivariational inequalities. The work is motivated by a mathematical model which describes static elastic contact problems with the Hooke constitutive law and nonmonotone subdifferential boundary conditions. The mechanical problem concerns a linear elastic body which may come in contact with a foundation. The dependence of the normal stress on the normal displacement is assumed to have nonmonotone character of the subdifferential form. We model the friction assuming that the tangential shear on the contact surface is given as a nonmonotone and possibly multivalued function of the tangential displacement. The contact problem under consideration is as follows.

Problem (P). *Find a displacement field $u\colon \Omega \to \mathbb{R}^d$ and a stress field $\sigma\colon \Omega \to \mathcal{S}_d$ such that*

$$
\begin{aligned}
\operatorname{div} \sigma(u) + f_0 = 0 &\quad \text{in} \quad \Omega, &(1)\\
\sigma(u) = \mathcal{A}\, e(u) &\quad \text{in} \quad \Omega, &(2)\\
u = 0 &\quad \text{on} \quad \Gamma_D, &(3)\\
\sigma n = f_N &\quad \text{on} \quad \Gamma_N, &(4)\\
-\sigma_N \in \partial j_N(x, u_N) &\quad \text{on} \quad \Gamma_C, &(5)\\
-\sigma_T \in \partial j_T(x, u_T) &\quad \text{on} \quad \Gamma_C, &(6)
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ represents a domain occupied by a linear elastic body, Γ_D, Γ_N and Γ_C are three open disjoint parts of the boundary $\partial\Omega$ such that meas$(\Gamma_D) > 0$, $\sigma(u)$ is the stress tensor, $e(u)$ is the linearized strain tensor, i.e. $e(u) = (e_{ij}(u))$, $e_{ij}(u) = \frac{1}{2}\,(u_{i,j} + u_{j,i})$, $\mathcal{A}$ is the elasticity tensor, n denoted a unit outward normal on $\partial\Omega$ and $\mathcal{S}_d$ is the linear space of second order symmetric tensors on $\mathbb{R}^d$. The densities of the body forces and of surface tractions are denoted by f_0 and f_N, respectively. We use the usual notation for the normal components and the tangential parts of vectors and tensors,

respectively, i.e. $u_N = u \cdot n$, $u_T = u - u_N n$, $\sigma_N = (\sigma n) \cdot n$ and $\sigma_T = \sigma n - \sigma_N n$. Here j_N and j_T are given functions and the notation ∂j_N and ∂j_T stands for the Clarke subdifferentials of $j_N(x, \cdot)$ and $j_T(x, \cdot)$.

Due to the nonmonotone character of multivalued boundary conditions (5) and (6), we formulate the problem in the form of a hemivariational inequality which involves the Clarke subdifferential of a locally Lipschitz functional. It should be mentioned that there is a large class of mechanical problems with nonconvex energy functions which are generally nonsmooth. For examples we refer to Panagiotopoulos [5] and Migórski and Ochal [4], and the references therein.

On the other hand in many problems of physics one has to solve boundary value problems in periodic media considering equations with highly oscillating coefficients. Quite often the size of the period is small compared to the size of a simple of the medium and an averaging process is needed to reduce the complexity of the problem. In the mathematical theory of homogenization, the problem is embedded into a sequence of similar problems and an asymptotic analysis is performed as the lengthscale goes to zero.

In this paper, first, we treat a hemivariational inequality which is the weak formulation of the model contact problem (P). We establish the existence of solutions to this hemivariational inequality. This result is a consequence of surjectivity result for multivalued operators, cf. [2]. Next, we deliver sufficient conditions under which the solution to the hemivariational inequality is unique. These results are quite general and they allow to deduce existence and uniqueness of solutions to other classes of elasticity models with nonmonotone and possible multivalued boundary conditions. Finally, using the notion of H-convergence of elasticity tensors $\mathcal{A}_\varepsilon$, we investigate the limit behavior, as $\varepsilon \to 0$, of the sequence of solutions to hemivariational inequalities. The limit hemivariational inequality is of the same form as (1)–(6) and corresponds to the homogenized tensor. We employ the notion of H-convergence adopted to the elasticity setting by Francfort and Murat [3] and Allaire [1].

The physical idea behind homogenization is to describe the macroscopic properties of media with highly heterogeneities of lengthscale ε modeled by tensors $\mathcal{A}_\varepsilon$ (for instance, composites with mixed periodically distributed different phases, fiber materials, stratified or porous media). From the mechanical point of view, the asymptotic analysis when $\varepsilon \to 0$ determines the large scale properties of the material without determining its fine scale structure. The limit homogenized tensor $\mathcal{A}$ defines an effective properties of the medium.

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