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Construction of Customized Mass-Stiffness Pairs Using Templates

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1 Introduction

Two standard procedures for building finite element mass matrices have been known and widely used since the mid 1960s, leading to consistent and diagonally-lumped forms. These models are denoted by $\mathbf{M}_C$ and $\mathbf{M}_L$, respectively, in the sequel. Collectively these take care of many engineering applications in structural dynamics. Occasionally, however, they fall short. The gap can be filled with a more general approach that relies on *templates*. These are algebraic forms that carry free parameters. This approach is covered in this paper using one-dimensional structural elements as expository examples.

The template approach has the virtue of generating a set of mass matrices that satisfy certain *a priori* constraints such as symmetry, nonnegativity, invariance and momentum conservation. In particular, the diagonally-lumped and consistent mass matrices can be obtained as instances. Thus those standard models are not excluded. Availability of free parameters, however, allows the mass matrix to be *customized* to special needs such as high precision in vibration analysis, or minimally dispersive wave propagation. This versatility will be evident from the examples. The set of parameters is called the *template signature,* and uniquely characterizes an element instance.

An attractive feature of templates for FEM programming is that each "custom mass matrix" need not be coded and tested individually. It is sufficient to implement the template as a single element-level module, with free parameters as arguments, and simply adjust the signature to the problem at hand. In particular the same module should be able to produce the conventional CMM and DLMM models, which can provide valuable crosschecking. The ability to customize the mass matrix is not free of cost. The derivation is more complicated, even for 1D elements, than those based on standard procedures. In fact, hand computations rapidly become unfeasible. Help from a computer algebra system (CAS) is needed to complete the task. When is this additional work justified? Two scenarios can be mentioned.

The first is *high fidelity systems*. Dynamic analysis covers a wide range of applications. There is a subclass that calls for a level of simulation precision beyond that customary in engineering analysis. Examples are deployment of precision structures, resonance analysis of machinery or equipment, adaptive active control, ultrasonics imaging, signature detection, radiation loss in layered circuits, and molecular- and crystal-level simulations in micro- and nano-mechanics. In structural static analysis an error of 20% or 30% in peak stresses is not cause for alarm — such discrepancies are usually covered adequately by safety factors. But a similar error in frequency analysis or impedance response of a high fidelity system may be disastrous. Achieving acceptable precision with a fine mesh, however, can be expensive. Model adaptivity comes to the rescue in statics; but this is less effective in dynamics on account of the time dimension. Customized elements may provide a practical solution: achieving adequate accuracy with a coarse regular mesh.

A second possibility is that the stiffness matrix comes from a method that *avoids displacement shape functions*. For example, assumed-stress or assumed strain elements. [Or, it could simply be an array of numbers provided by a black-box program, with no documentation explaining its source.] Under this scenario the concept of "consistent mass matrix," in which velocity shape functions are taken to coincide with displacement ones, loses its comfortable variational meaning. One way out is to take the mass matrix of an element with similar geometry and freedom configuration derived with shape functions, and to pair it with the given stiffness. But in certain cases, notably when the FEM model has rotational freedoms, this may not be easy or desirable.

2 Parametrization Techniques

There are several ways to parametrize mass matrices. Three techniques found effective in practice are summarized below. All of them are illustrated in the worked out examples in [1].

Matrix-Weighted Parametrization. A matrix-weighted mass template for element e is a linear combination of $(k+1)$ component mass matrices, $k \geq 1$ of which are weighted by parameters:

$$\mathbf{M}^e \stackrel{\text{def}}{=} \mathbf{M}_0^e + \mu_1 \mathbf{M}_1^e + \ldots \mu_k \mathbf{M}_k^e \tag{1}$$

Here $\mathbf{M}_0^e$ is the *baseline mass matrix*. This should be an acceptable mass matrix on its own if $\mu_1 = \ldots \mu_k = 0$. The simplest instance of (1) is a linear combination of the consistent and diagonally-lumped masses: $\mathbf{M}^e \stackrel{\text{def}}{=} (1-\mu)\mathbf{M}_C^e + \mu \mathbf{M}_L^e$. This can be reformatted as (1) by writing $\mathbf{M}^e = \mathbf{M}_C^e + \mu(\mathbf{M}_L^e - \mathbf{M}_C^e)$. Here $k = 1$, the baseline is $\mathbf{M}_0^e \equiv \mathbf{M}_C^e$, $\mu \equiv \mu_1$ and $\mathbf{M}_1^e$ is the "consistent mass deviator" $\mathbf{M}_L^e - \mathbf{M}_C^e$.

A matrix-weighted mass template represents a tradeoff. It cuts down on the number of free parameters, which is essential for 2D and 3D elements. It makes it easier to satisfy conservation and nonnegativity conditions through appropriate choice of the $\mathbf{M}_i^e$. On the minus side it generally spans only a subspace of acceptable matrices.

Spectral Parametrization. This has the form

$$\mathbf{M}^e \stackrel{\text{def}}{=} \mathbf{H}^T \mathbf{D}_\mu \mathbf{H}, \qquad \mathbf{D}_\mu = \textbf{diag}\,[\, c_0\mu_0 \;\; c_1\mu_1 \;\; \ldots \;\; c_k\mu_k \,]\,. \tag{2}$$

in which $\mathbf{H}$ is a generally full matrix. Parameters $\mu_0 \ldots \mu_k$ appear as entries of the diagonal matrix $\mathbf{D}_\mu$. Scaling coefficients c_i may be introduced for convenience. Some of the μ coefficients may be preset from *a priori* conservation conditions.

Configuration (2) occurs naturally when the mass matrix is constructed first in generalized coordinates, followed by transformation to physical coordinates via $\mathbf{H}$. If the generalized mass is derived using mass-orthogonal functions (for example, Legendre polynomials in 1D elements), the unparametrized generalized mass $\mathbf{D} = \textbf{diag}\,[\, c_0 \;\; c_1 \;\; \ldots c_k \,]$ is diagonal. Parametrization is effected by scaling entries of this matrix. Some entries may be left fixed, however, to satisfy *a priori* constraints. Expanding (2) and collecting matrices that multiply μ_i leads to a matrix weighted combination form (1) in which each $\mathbf{M}_i^e$ is a rank-one matrix. The analogy with the spectral representation theorem of symmetric matrices is obvious. But in practice it is usually better to work directly with the congruential representation (2).

Entry-Weighted Parametrization. An entry-weighted mass template applies parameters directly to every entry of the mass matrix, except for *a priori* constraints on symmetry, invariance and conservation. This form is the most general one and can be expected to lead to best possible solutions. But it is restricted to simple (usually 1D) elements because the number of parameters grows quadratically in the matrix size, whereas for the other two schemes it grows linearly.

Combined Approach. A hierarchical combination of parametrization schemes can be used to advantage if the kinetic energy can be naturally decomposed from physics. For example the Timoshenko beam element studied in [1] uses a two-matrix-weighted template as top level. The two components are constructed by spectral and entry-weighted parametrization, respectively.

References

[1] C. A. Felippa, Construction of customized mass-stiffness pairs using templates, invited contribution to Special Issue in honor of A. K. Noor, *ASCE J. Aerospace*, **19:4**, 241–258, 2006.