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SAINT VENANT'S PRINCIPLE AND ITS IMPLICATIONS TO TOPOLOGY OPTIMIZATION

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ABSTRACT

One approach to topology optimization consists in considering an infinitesimal hole in a given body and study the perturbation induced in the solution of a certain elliptic equation, then try to describe the infinitesimal variation of a given objective function. When one studies the difference between the two solutions of the elliptic equation (the perturbed solution minus the unperturbed one), it is interesting to note that this difference satisfies the hypotheses of Saint Venant's principle: the forces associated to it are zero outside the infinitesimal hole, and the forces appearing in the hole are balanced (they have zero resultant and zero momentum).

Saint Venant's principle is an old and well-known statement in continuum mechanics. It is often referred to but not always well understood, rarely stated with precision and hardly ever proven with mathematical rigour. Saint Venant's principle is usually stated in a rather vague form: A local distribution of forces in a small region of a body has little effect on the deformation of the body far from that region – the only relevant quantities are the resultant force and the resultant momentum.

In 1967, Knowles considered in [1] the case of a scalar elliptic equation in a two-dimensional domain of the shape of a curved beam, where he formulated and proved in full mathematical rigour Saint Venant's principle. This is the only rigourous proof of the principle, to the extent of the author's knowledge. Knowles considered a zero Dirichlet boundary condition at one end of the "beam", a zero Neumann boundary condition along the two sides. At the other end of the "beam", the Neumann data has zero integral. Note that, unlike in the elasticity setting, for a scalar elliptic equation the momentum of the applied "forces" is not defined. Knowles proves that the energy stored in the "beam" decreases exponentially with the distance to the extremity.

In the present communication we prove Saint Venant's principle for a scalar elliptic equation, in a domain of arbitrary shape, in any space dimension. The "forces" are applied only in a small ball of radius ρ and have zero integral. We prove that the energy stored at a fixed distance from the ball decreases, as $\rho \to 0$, with the speed of a certain power of ρ. Auxiliary results, such as Poincaré-Wirtinger inequality for functions defined on a sphere and a result from spectral geometry about the second eigenvalue of the Laplace operator on a sphere, are employed.

We point out several links between Saint Venant's principle and the topology derivative. We describe an asymptotic development of the perturbed solution, independently of the objective functional. In three dimensions, the estimate in the asymptotic development can be improved (by appling Saint Venant's principle) in any region distant enough from the hole.

By an argument related to structural optimization, we show that, for arbitrarily shaped domains, exponential decay of the energy is not to be expected in Saint Venant's principle.

REFERENCES

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