**8th. World Congress on Computational Mechanics (WCCM8)**
**5th European Congress on Computational Methods in Applied Sciences and Engineeering (ECCOMAS 2008)**
**June 30 –July 5, 2008**
**Venice, Italy**

# Plane Wave Discontinuous Galerkin Methods

*** R. Hiptmair[1], C. Gittelson[2], and I. Perugia[3]**

[1] SAM, ETH Zürich
CH-8092 Zürich
hiptmair@sam.math.ethz.ch

[2] SAM, ETH Zürich
CH-8092 Zürich
giclaude@student.ethz.ch

[3] Dipartimento di Matematica, Universitá di Pavia,
Via Ferrata 1, I-27100 Pavia,
ilaria.perugia@unipv.it

**Key Words:** *Wave propagation, finite element methods, discontinuous Galerkin methods, plane waves, ultra weak variational formulation, duality estimates*

## ABSTRACT

Standard low order Lagrangian finite element discretization of boundary value problems for the Helmholtz equation $-\Delta u - \omega^2 u = f$ are afflicted with the so-called pollution phenomenon: though for sufficiently small $h\omega$ an accurate approximation of $u$ is possible, the Galerkin procedure fails to provide it. Attempts to remedy this have focused on incorporating extra information in the form of plane wave functions $\boldsymbol{x} \mapsto \exp(i\omega \boldsymbol{d} \cdot \boldsymbol{x})$, $|\boldsymbol{d}| = 1$, into the trial spaces. Prominent examples of such methods are the plane wave partition of unity finite element method of Babuska and Melenk [1], and the ultra-weak Galerkin discretization due to Cessenat and Despres [3, 4]. Both perform well in computations, see the articles by Monk and Hutunen [8, 7, 6] for computational results for the ultra-weak approach.

It turns out that the latter method can be recast as a special so-called discontinuous Galerkin (DG) method employing local trial spaces spanned by a few plane waves. In a sense, this is a special case of a Trefftz-type approximation. This perspective paves the way for marrying plane wave approximation with many of the various DG methods developed for 2nd-order elliptic boundary value problems. We have pursued this for a class of primal mixed DG methods, which generalize the ultra-weak scheme.

For these methods we have developed a convergence analysis for the $h$-version, which achieves convergence through mesh refinement [5]. Key elements are approximation estimates for plane waves and sophisticated duality techniques. The latter entail estimating how well local plane waves can approximate the solution of a dual problem. Unfortunately, we cannot help invoking general polynomial estimates in Sobolev spaces for this purpose. This results in a minimal resolution condition that the trial space has to meet in order to yields quasi-optimal DG solutions. Thus, the notorious pollution effect that haunts local discretizations of wave propagation problems manifests itself in the theoretical estimates [2].

Only in 1D the pollution effect can be avoided by the plane wave DG methods. Conversely, in higher dimensions only a few of infinitely many wave directions will be represented in the trial space, usually equidistributed on unit circle/unit sphere. If the solution of a boundary value problem contains plane wave components falling in between these directions, *numerical dispersion* is inevitable. Stark evidence is offered by numerical experiments in 2D.

## References


[1] I. Babuška and J. Melenk, *The partition of unity method*, Int. J. Numer. Methods Eng., 40 (1997), pp. 727–758.

[2] I. Babuška and S. Sauter, *Is the pollution effect of the FEM avoidable for the Helmholtz equation?*, SIAM Review, 42 (2000), pp. 451–484.

[3] O. Cessenat and B. Despres, *Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz equation*, SIAM J. Numer. Anal., 35 (1998), pp. 255–299.

[4] ——, *Using plane waves as base functions for solving time harmonic equations with the ultra weak variational formulation*, J. Computational Acoustics, 11 (2003), pp. 227–238.

[5] C. Gittelson, R. Hiptmair, and I. Perugia, *Plane wave discontinuous galerkin methods*. In preparation.

[6] T. Huttunen, M. Malinen, and P. Monk, *Solving maxwell's equations using the ultra weak variational formulation*, J. Comp. Phys., (2006).

[7] T. Huttunen and P. Monk, *The use of plane waves to approximate wave propagation in anisotropic media*, J. Computational Mathematics, 25 (2007), pp. 350–367.

[8] T. Huttunen, P. Monk, and J. Kaipio, *Computational aspects of the ultra-weak variational formulation*, J. Comp. Phys., 182 (2002), pp. 27–46.