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Uncertainty quantification for conservation laws

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ABSTRACT

We are interested in application of UQ (Uncertainty Quantification) with Polynomial Chaos (PC) technics for complex compressible flows such as the ones encountered in ICF (Inertial Confinement Fusion); the general frame is uncertain systems of conservation laws:

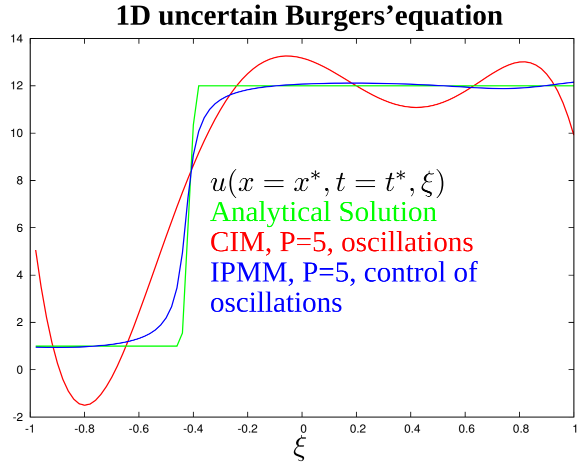
$$\partial_t u(x,t,\xi) + \partial_x f(u(x,t,\xi)) = 0 \text{ with } u(x,t,\xi) \in \mathbb{R}^n, \text{ where } \xi \text{ is the random vector.}$$

A basic PDE model is compressible Euler equations. In this work, we discuss a simpler example (stochastic inviscid Burgers'equation) to describe the new method we have developped to control Gibbs phenomenon (arising when the solution exhibits steep dependencies with respect to the random variable)still working on the initial polynomial basis.

We begin by a short review: PC methods were first introduced by Ghanem and Spanos [GS91] and are based on the Homogeneous Chaos Theory of Wiener [Wie38]. They appeared to be a good alternative to statistical methods (as Monte Carlo simulations and its modifications) for UQ as these latters can become too expensive due to the high number of samples required and time consuming codes. PC methods were successfully used for accurately solving many problems (stochastic elastic materials [GS91], finite deformations, heat conduction [XK03], incompressible flows [OLM07], reacting flows and detonation [DP07]...) and for giving rich statistical information through the polynomial coefficients (stochastic moments, Sobol sensibilities, pdf,...). However, classical approaches fail to approximate the solution in the case of ”complex” flows implying for example discontinuities with respect to the random variable (see [OLMG04] and [Cho74]). Conservation laws, known to generate shocks, can give birth to those kinds of difficulties. Several directions have been investigated in order to treat the Gibbs phenomenon due to polynomial order truncation as the use of Haar wavelets [OLMG04] or adaptative methods as ME-GPC [WK06]. All these methods rely on a discretisation of the random space: in the case of a moving discontinuity, with adaptative methods, the number of random subdomains can quickly become important (each time step needs a new refinement in the random space). Besides, they need interface tracking technics which are simple for one dimensional problems but are known to become quite tricky in higher dimensions.

To overcome these issues, we have developped a new method (IPMM for Intrusive Polynomial Moment Method) which is based on a theoretical parallel between Classical PC (CIM for Classical Intrusive Method) and ERT (Extended Rational Thermodynamics) ([MR98], [CLL94]). In ERT, the main variable v is the so-called *entropy variable* defined for a system of conservation laws through an entropy-entropy flux pair (s, g) (ensureing the hyperbolicity of the system). In our approach, this variable becomes the main variable in the polynomial expansion $v(x, t, \xi) = \nabla_u s(u(x, t, \xi)) \approx \sum_{i=0}^{P} v_i(x, t)\phi_i(\xi)$ (where $(\phi_i)_{i\in\mathbb{N}}$ is the orthogonal polynomial basis). Several properties can then be proved for the new system as hyperbolicity (well-posedness), minoration and majoration of eigenvalues (control of the CFL condition) and minimisation of entropy. In the case the entropy pair is not unique for the initial system, particular choices can lead to a better control of oscillations.

We illustrate the new IPMM the stochastic inviscid Burgers'equation in one dimensional random dimension corresponding to a uncertain shock position (figure below). We will also present relevant test-cases as succesions of shocks with stochastic forces or stochastic interface position problems which are representative of physical difficulties encountered in FCI to valid our method.



The new IPMM system for Burgers'equation:

$$\partial_t \int u\Big(\sum_{i=0}^{P} v_i\phi_i\Big) \begin{pmatrix} \phi_0 \\ \dots \\ \phi_P \end{pmatrix} \mathrm{d}w$$

$$+\partial_x \int \frac{u^2\Big(\sum_{i=0}^{P} v_i\phi_i\Big)}{2} \begin{pmatrix} \phi_0 \\ \dots \\ \phi_P \end{pmatrix} \mathrm{d}w = 0$$

where $\mathrm{d}w$ is the probability measure.

The entropy: $s(u) = -\ln(u - u_-) - \ln(u_+ - u)$, with $(u_-, u_+) = (0.5, 12.5)$.

We will finally present another application of the IPMM on compressible gas dynamics equations in Eulerian and Lagrangian coordinates and we emphasize the fact that our resulting schemes are conservative for all variables.

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