8th. World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2008)
June 30 –July 5, 2008
Venice, Italy

A NEW PETROV-GALERKIN FINITE ELEMENT METHOD FOR STABILIZING REACTION-DIFFUSION EQUATIONS

*** Florin Ilinca[1] and Jean-François Hétu[2]**

[1] National Research Council
75 de Mortagne, Boucherville,
Qc, J4B 6Y4, Canada
florin.ilinca@cnrc-nrc.gc.ca

[2] National Research Council
75 de Mortagne, Boucherville,
Qc, J4B 6Y4, Canada
jean-francois.hetu@cnrc-nrc.gc.ca

Key Words: *Stabilized finite elements, Petrov-Galerkin method, reaction-diffusion equation, Taylor series expansion.*

ABSTRACT

This paper presents the Source Stabilized Petrov-Galerkin (SSPG) method for solving reaction-diffusion problems described by the modified Helmholtz operator. This new stabilized finite element formulation improves the accuracy over the Galerkin, GGLS [1] and PGEM [2] methods. The method is shown to be of the Petrov-Galerkin type and consists in modifying the weighting function. The stabilized formulation is also shown to be equivalent to a modified equation solved by the Galerkin method. The modified partial differential equation is obtained from the first order Taylor series expansion around mesh nodes of the terms contained in the original equation. The performance of the SSPG method is illustrated on one- and two-dimensional problems and the results are compared with those provided by the Galerkin, GGLS, PGEM and the mass lumping of the source term (MLST). For the one-dimensional case and a uniform mesh the new method yields the exact nodal solution as the GGLS formulation. The SSPG is shown to perform better than the other methods in the two-dimensional case.

Lets consider the model problem: $\sigma^2 u - \epsilon^2 \Delta u - f = 0$ on Ω.
Therefore the stabilized SSPG formulation is: *Find* $u_h \in V_h$ *such that*

$$\begin{aligned} &\int_\Omega \sigma^2 u_h N_I \, \mathrm{d}\Omega + \int_\Omega \epsilon^2 \nabla u_h \cdot \nabla N_I \, \mathrm{d}\Omega - \int_\Omega f N_I \, \mathrm{d}\Omega \\ - &\int_\Omega \xi_I \nabla \left(\sigma^2 u_h - \epsilon^2 \Delta u_h - f\right) \cdot (\mathbf{x} - \mathbf{x}_I) N_I \, \mathrm{d}\Omega = \int_{\Gamma_q} q N_I \, \mathrm{d}\Gamma, \quad \forall N_I \in V_h^0, \end{aligned} \tag{1}$$

where

$$\xi_I = \frac{\cosh(\sqrt{6\alpha_I}) + 2}{\cosh(\sqrt{6\alpha_I}) - 1} - \frac{1}{\alpha_I}, \quad \alpha_I = \frac{\sigma^2 h_I^2}{6\epsilon^2}. \tag{2}$$

The SSPG equation can also be formulated as a Petrov-Galerkin method in the form:

$$\begin{aligned} &\int_\Omega \sigma^2 u_h N_I \, \mathrm{d}\Omega + \int_\Omega \epsilon^2 \nabla u_h \cdot \nabla N_I \, \mathrm{d}\Omega - \int_\Omega f N_I \, \mathrm{d}\Omega \\ + &\sum_K \int_{\Omega_K} \xi_I \left(\sigma^2 u_h - \epsilon^2 \Delta u_h - f\right) \left[n_d N_I + (\mathbf{x} - \mathbf{x}_I) \cdot \nabla N_I\right] \mathrm{d}\Omega = \int_{\Gamma_q} q N_I \, \mathrm{d}\Gamma. \end{aligned} \tag{3}$$

where n_d is the dimension of the problem ($n_d = 1$, 2 or 3).

Results are shown here for a test problem having the analytical solution

$$u(x,y) = 1 - \frac{\sinh\left(\frac{\sigma}{\epsilon}(1-x)\right)}{2\sinh\left(\frac{\sigma}{\epsilon}\right)} - \frac{\sinh\left(\frac{\sigma}{\epsilon}(1-y)\right)}{2\sinh\left(\frac{\sigma}{\epsilon}\right)}, \quad \text{for } 0 \le x, y \le 1. \tag{4}$$

Dirichlet conditions are imposed on the boundaries. The mesh is shown in Fig. 1 and the exact nodal solution is illustrated in Fig. 2 for $\sigma^2 = 1$, $\epsilon^2 = 10^{-8}$ and $f = 1$. Solution errors with respect to the exact solution at $P(x = 0.05, y = 0.05)$ when varying ϵ^2 are shown in Fig. 3. The error of the Galerkin method is much higher than that of the other methods and hence it was not included. For very low and for very high diffusion coefficient all stabilized methods perform well. Discrepancies are observed for ϵ^2 between 10^{-6} and 10^{-2} when the appropriate balance between the natural diffusion and the anti-diffusive contribution of the source term need to be reached. The results indicate that PGEM overestimates the exact solution (not enough diffusion), whereas the MLST solution is over-diffusive. The mean nodal error is shown in Fig. 4 for the various finite element solutions. The SSPG method leads to the most accurate solution for all values of the diffusion coefficient.

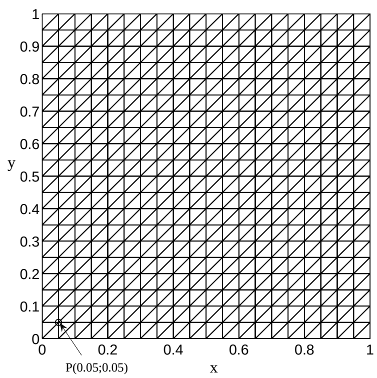

Figure 1. Mesh for two-dimensional problem

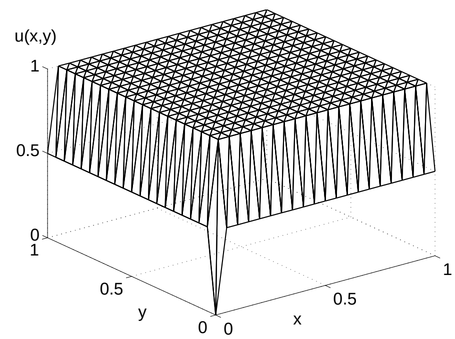

Figure 2. Exact nodal solution for $\epsilon^2 = 10^{-8}$

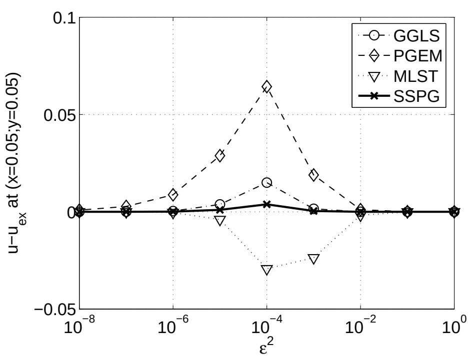

Figure 3. Solution errors at $P(x = 0.05, y = 0.05)$

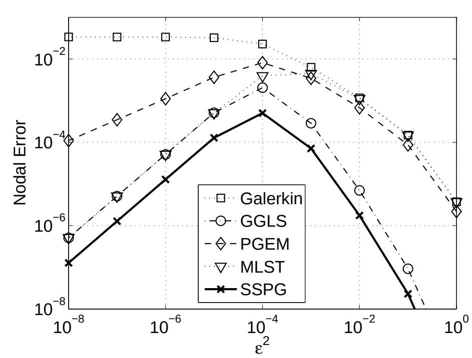

Figure 4. Mean nodal error

REFERENCES

[1] L.P. Franca and E.G. Dutra Do Carmo, "The Galerkin gradient least-squares method," *Comput. Methods Appl. Mech. Engrg.*, vol. 74, pp. 41–54, 1989.

[2] L.P. Franca, A.L. Madureira and F. Valentin, "Towards multiscale functions: enriching finite element spaces with local but not bubble-like functions," *Comput. Methods Appl. Mech. Engrg.*, vol. 194, pp. 3006–3021, 2005.