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THE STIFFEST PLATES AND SHELLS OF THE UNIFORMLY DISTRIBUTED KELVIN MODULI

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ABSTRACT

The problem of optimal choice of the elastic material at each point of the structure to maximize its global stiffness was formulated and solved for the first time in paper by Bendsøe et al.[1] for the linear elasticity case. The isoperimetric condition was expressed in terms of an integral of the norm of the Hooke tensor, directly depending on Kelvin's moduli. In this method no additional conditions on the Kelvin moduli (or eigenvalues of the Hooke tensor) were imposed. Generalization of this method to shells was the subject of the paper by Gaile et al.[3]. It is thought appropriate, however, to fix the values of the Kelvin moduli within the whole design domain while allowing for variation of the geometrical parameters of the underlying microstructure. In the plane case Hooke's tensor is determined by three Kelvin's moduli λ_i, i=1,2,3, and by three angles φ_i determining three eigentensors $\boldsymbol{\omega}_i$, see [2,4]. One of the angles shows only an angle of observation, while two remaining are called distributors of stiffnesses, see Rychlewski [4, Sec.9]. The problem of optimal layout of the angles φ_i in plates loaded in-plane or in bending plates at fixed values of λ_i is not difficult and will be treated here as a particular case of the problem of coupling of the membrane and bending states within a plate or within a shell. In such a coupled problem the energy density is expressed by the formula involving both kinds of deformation $W = (\boldsymbol{\varepsilon} \cdot \mathbf{A}\boldsymbol{\varepsilon} + \boldsymbol{\kappa} \cdot \mathbf{A}\boldsymbol{\kappa})/2$ where $\mathbf{A}$ represents the membrane stiffness tensor, $\boldsymbol{\varepsilon} = (\varepsilon_{\alpha\beta})$ represents the membrane deformation tensor, while $\kappa_{\alpha\beta} = (h/\sqrt{12})\rho_{\alpha\beta}$, where $\rho_{\alpha\beta}$ are changes of curvature; h stands for the shell or plate thickness and the *dot* implies the full contraction. Let us consider the spectral decomposition of the stiffness tensor: $\mathbf{A} = \lambda_1\boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_1 + \lambda_2\boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_2 + \lambda_3\boldsymbol{\omega}_3 \otimes \boldsymbol{\omega}_3$, where $\lambda_1 > \lambda_2 > \lambda_3$. The layout of the tensor fields $\boldsymbol{\omega}_i$ on the middle surface will be optimized to make the shell as stiff as possible. The elastic potential W will be viewed as a function of $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$ and $\boldsymbol{\omega}_i$. The key problem

reduces to solving the local optimization problem: $U = \max\{ W(\boldsymbol{\varepsilon},\boldsymbol{\kappa},\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\omega}_3) \mid \boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_j = \delta_{ij} \}$. This problem can be solved explicitly; the effective potential assumes the form

$$4U(\boldsymbol{\varepsilon},\boldsymbol{\kappa}) = (\lambda_1+\lambda_2)\left(\|\boldsymbol{\varepsilon}\|^2+\|\boldsymbol{\kappa}\|^2\right)+(\lambda_1-\lambda_2)\left[\left(\|\boldsymbol{\varepsilon}\|^2-\|\boldsymbol{\kappa}\|^2\right)^2+(2\boldsymbol{\varepsilon}\cdot\boldsymbol{\kappa})^2\right]^{1/2}$$

where $\|\boldsymbol{\varepsilon}\|^2 = \boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}$. One can indicate the tensors $\boldsymbol{\omega}_i$ realizing maximum, these formulae will be not displayed here. Eventually, the final explicit representation of the optimal Hooke tensor **A** has been constructed. To solve the global optimization problem one should: solve the equilibrium problem for the shell (or the plate) characterized by the following effective hyperelastic constitutive equations:

$$\mathbf{N} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}},\ \mathbf{M} = \frac{h}{\sqrt{12}}\frac{\partial U}{\partial \boldsymbol{\kappa}},$$

at given shell geometry, loading and boundary conditions. Upon determining the deformation fields at each point of the middle surface one should determine the tensor **A** according to the spectral representation involving tensors $\boldsymbol{\omega}_i$ expressed in terms of the deformations. In the uncoupled case, e.g. when the bending deformations vanish, the potential U assumes the form well known from Refs.[1-3].

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