8th. World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineeering (ECCOMAS 2008)
June 30 –July 5, 2008
Venice, Italy

OPTIMAL DESIGN OF SHALLOW SHELLS BY THE TRANSLATION METHOD

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Key Words: *Shallow Shells, Minimal Compliance, Translation Method.*

ABSTRACT

The question of minimizing the compliance of an elastic structure by optimal distribution of two isotropic materials within its volume was fully solved in 1980s for thin plate and plane stress problems, see [1, 2, 3]. In both cases, the procedure of analysis consists of three steps. Firstly, the problem is well-posed (relaxed) by allowing microstructural mixtures in the optimal solution. Secondly, relaxed complementary energy of the structure is estimated from below by the so-called translation method. Proving attainability of the translation estimation on certain microstructures completes the algorithm. In the sequel we follow the second step according to above sketched pattern in seeking solution of the relaxed minimal compliance problem of a shell, where bending and membrane effects are coupled. To deal with this phenomenon, we make use of the generalized translation method of Milton, see [4], with the same translation functional as in separately treated thin plate and plane stress cases. Let $\left(\xi^1, \xi^2\right) \equiv \xi \in \Omega \subset \mathbb{R}^2$ parameterize the mid-surface S of a thin shell of constant thickness h. The shell is made of two isotropic well-ordered materials whose constitutive properties are given by Kelvin and Kirchhoff moduli $k_\alpha, \mu_\alpha,\ \alpha = 1, 2,\ k_1 < k_2, \mu_1 < \mu_2$ and its inverses $K_\alpha = \frac{1}{k_\alpha}, L_\alpha = \frac{1}{\mu_\alpha}$. Both materials are homogeneously distributed over the thickness of the shell and their total amounts are fixed by the condition $\frac{1}{|\Omega|} \int_\Omega \theta(\xi) \sqrt{g}\, d\xi = m$, where $\theta \in L^\infty\left(\Omega, [0, 1]\right)$ denotes a function describing the density of material 2. in the composite and g stands for the determinant of a metric tensor defined on S. Let $\mathfrak{S}$ denote the compliance tensor of a shell and set $\boldsymbol{\sigma} = \left[\mathbf{N}, \mathbf{M}\right]^\top$, where $\mathbf{N}, \mathbf{M}$ stand for membrane forces and bending moments tensors. In the relaxed formulation of the minimum compliance problem we seek so-called translation parameters α, β, γ giving the function $W_t^* = \frac{1}{2}\boldsymbol{\sigma}^\top : \mathfrak{S}\left(\theta, \alpha, \beta, \gamma\right) : \boldsymbol{\sigma}$ its maximal value over the set $\mathcal{Z} = \{(\alpha, \beta, \gamma) : -K_2 \leq \alpha \leq L_2,$ $0 \leq \beta \leq L_2, \gamma^2 \leq \left(K_2 + \alpha\right)\left(K_2 + \beta\right), \gamma^2 \leq \left(L_2 - \alpha\right)\left(L_2 - \beta\right)\}$ while θ is treated as a parameter, see [5]. By Castigliano Theorem, minimizing thus obtained optimal complementary energy functional amounts to calculating the minimum value of the compliance of the shell, see e.g. [3]. Optimal values of α, β, γ locally depend on θ and five invariants $tr\mathbf{N}, tr\mathbf{M}, tr\mathbf{N}^2, tr\mathbf{M}^2, tr\left(\mathbf{N} \cdot \mathbf{M}\right)$. We simplify our task by assuming that optimal values of α, β are equal to those obtained from case-by-case analysis of pure membrane and bending problems whereas γ is determined by formulas defining $\mathcal{Z}$. Hence, we reduce the set of invariants to $\zeta_N = \frac{|tr\mathbf{N}|}{\sqrt{2}\|dev\mathbf{N}\|}, \zeta_M = \frac{|tr\mathbf{M}|}{\sqrt{2}\|dev\mathbf{M}\|}$.

For $\alpha_2 = \frac{\Delta K[L]_\theta - \Delta L[K]_\theta \zeta_N}{\Delta K + \Delta L \zeta_N}$, $\beta_2 = \frac{\Delta K[L]_\theta - \Delta L[K]_\theta \zeta_M}{\Delta K + \Delta L \zeta_M}$, $\zeta_N^1 = \frac{K_2 + [L]_\theta}{\theta \Delta L}$, $\zeta_N^2 = \frac{\theta \Delta K}{L_2 + [K]_\theta}$, $\zeta_M^1 = \frac{[L]_\theta \Delta K}{[K]_\theta \Delta L}$, $\zeta_M^2 = \zeta_N^2$, $\Delta F = |F_1 - F_2|$, $[F] = \theta F_1 + (1-\theta) F_2$ we have 9 regions of optimality, see Table 1. Consequently, thus obtained tensor $\mathfrak{S}$ nonlinearly depends on $\mathbf{N}$ and $\mathbf{M}$, see [5], but it is isotropic.

	$0 \le \zeta_N \le \zeta_N^2$	$\zeta_N^2 \le \zeta_N \le \zeta_N^1$	$\zeta_N^1 \le \zeta_N$
$\zeta_M^1 \le \zeta_M$	$\alpha = L_2$, $\beta = 0$, $\gamma = 0$	$\alpha = \alpha_2$, $\beta = 0$ $\gamma = \pm \min\left\{ \sqrt{K_2(K_2+\alpha)}, \sqrt{L_2(L_2-\alpha)} \right\}$	$\alpha = -K_2$, $\beta = 0$, $\gamma = 0$
$\zeta_M^2 \le \zeta_M \le \zeta_M^1$	$\alpha = L_2$, $\beta = \beta_2$, $\gamma = 0$	$\alpha = \alpha_2$, $\beta = \beta_2$, $\gamma = \pm \min\left\{ \sqrt{(K_2+\alpha)(K_2+\beta)}, \sqrt{(L_2-\alpha)(L_2-\beta)} \right\}$	$\alpha = -K_2$, $\beta = \beta_2$, $\gamma = 0$
$0 \le \zeta_M \le \zeta_M^2$	$\alpha = L_2$, $\beta = L_2$, $\gamma = 0$	$\alpha = \alpha_2$, $\beta = L_2$, $\gamma = 0$	$\alpha = -K_2$, $\beta = L_2$, $\gamma = 0$

Table 1: Optimal values of α, β, γ

The above described method leads to an approximate value of the lower bound on the compliance of a shell. The question of proper description of optimal microstructures is still open in the context of shells, however some attempts are worth pointing out, see e.g. [6, 7]. An approach based on invariant description of the Hooke's tensors of sequential laminates is being currently researched by the Author.

ACKNOWLEDGEMENTS

The paper was prepared within the Research Grant no 4 T07A 038 30 financed by the Polish Ministry of Science and Higher Education: *Theory and numerical implementation of the relaxed formulations of optimization problems involving coupled fields. Designing of the material layout within composite structures*.

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