8th. World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineeering (ECCOMAS 2008)
June 30 –July 5, 2008
Venice, Italy

EFFECT OF ELEMENT DISTORTION ON THE NUMERICAL DISPERSION OF SPECTRAL ELEMENT METHODS

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Key Words: *Dispersion Analysis, Spectral Element Method, Wave Equation.*

ABSTRACT

Spectral elements are well established in the field of computational seismology, specially because they inherit the flexibility of finite element methods and have low numerical dispersion error. The latter is experimentally acknowledged, but is theoretically known in limited cases, such as Cartesian meshes [2]. It is well known that a finite element mesh can contain distorted elements that generate numerical errors for very large distortion. In the present work we study the effect of element distortion on the numerical dispersion error and determine the distortion range in which an accurate solution is obtained with a given error tolerance.

We consider spectral elements for the 2D acoustic wave equation $\ddot{u}(\boldsymbol{x},t) = c^2 \Delta u(\boldsymbol{x},t)$. Let us write its semi-dicretization in space in the form $\boldsymbol{M}\ddot{\boldsymbol{u}}^*(t) + c^2 \boldsymbol{K}\boldsymbol{u}^* = \boldsymbol{0}$, where $u_p^*(t) \approx u(\boldsymbol{x}_p, t)$ and $\boldsymbol{M}, \boldsymbol{K}$ are the mass and stiffness matrices, respectively. Plugging into this system a harmonic plane wave $\boldsymbol{u}^*(t) = \exp(-i\omega^* t)\boldsymbol{w}$, $w_p = \exp(i\boldsymbol{\kappa} \cdot \boldsymbol{x}_p)$, we find

$$\boldsymbol{K}\boldsymbol{w} = \chi \boldsymbol{M}\boldsymbol{w}, \quad \chi = (\omega^*/c)^2 . \tag{1}$$

This is an over-constrained system of equations for χ. We approximate χ in the least-squares sense by the Rayleigh quotient

$$\chi^* = \frac{\overline{\boldsymbol{w}}^T \boldsymbol{K}\boldsymbol{w}}{\overline{\boldsymbol{w}}^T \boldsymbol{M}\boldsymbol{w}}, \text{ i.e.,} \quad \omega^* = c\sqrt{\frac{\overline{\boldsymbol{w}}^T \boldsymbol{K}\boldsymbol{w}}{\overline{\boldsymbol{w}}^T \boldsymbol{M}\boldsymbol{w}}}. \tag{2}$$

Assuming an unbounded, periodic mesh formed by the repetition of a patch of N_e elements (as in Figure 1), we have that (2) reduces to

$$\chi^* = \frac{\sum_{e=1}^{N_e} \overline{\boldsymbol{w}_e}^T \boldsymbol{K}_e \boldsymbol{w}_e}{\sum_{e=1}^{N_e} \overline{\boldsymbol{w}_e}^T \boldsymbol{M}_e \boldsymbol{w}_e}, \tag{3}$$

where $\boldsymbol{M}_e$, $\boldsymbol{K}_e$, and $\boldsymbol{w}_e$ are element restrictions of the global arrays $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{w}$. For Cartesian meshes, expression (3) reduces to Rayleigh quotients of element matrices from 1D problems. The estimate (3) is similar in fully-discrete schemes (e.g., $\chi = (2\sin(\omega^*\Delta t/2)/\Delta t)^2$ for the leapfrog scheme).

Let $\boldsymbol{\kappa} = \kappa(\cos\theta, \sin\theta)$. We evaluate κ from the number of grid points per wavelength

$$G = \frac{\lambda}{\Delta x} = \frac{2\pi}{|\boldsymbol{\kappa}|\,\Delta x}, \quad \Delta x = \frac{1}{N}\sqrt{\frac{1}{N_e}\sum_{e=1}^{N_e}|\Omega_e|}, \tag{4}$$

where N is the polynomial degree and $|\Omega_e|$ the area of the e-th element of the patch that generates the mesh. Note that Δx corresponds to the standard definition for square elements.

We used the dispersion estimate (3) to study the dispersion of spectral elements in several meshes, in particular the periodic mesh defined in [1], summarized in Figure 1. The central node of the patch moves along the dashed line so that the internal angle of the upper corner of the element 1 varies from 90^o (*square*) to 180^o (*square degenerated to a triangle*). Figure 2 shows the variation with α of the phase error $(c^* - c)/c$, $c^* = \omega^*/\kappa$, considering Chebyshev and Legendre collocation points with $G = 5$ and $N = 8$. The Chebyshev spectral element uses an optimal blended operator [2]. We see that both methods have a low sensitivity (for a large range of the element distortion) to the mesh deformation at this refinement level, confirming the high accuracy of spectral element methods, even in distorted grids.

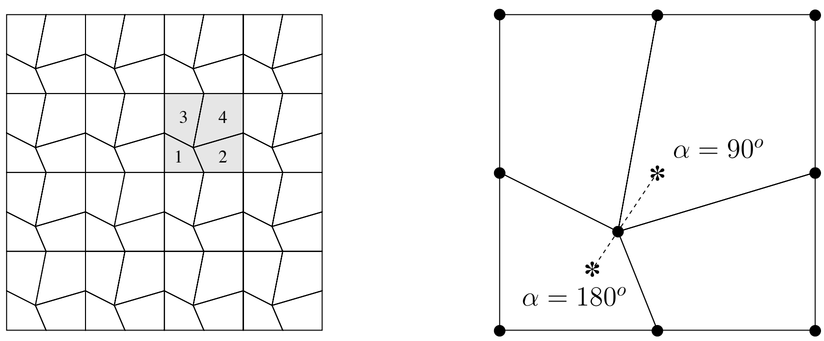

Figure 1: Mesh generated by a patch of four non-rectangular elements.

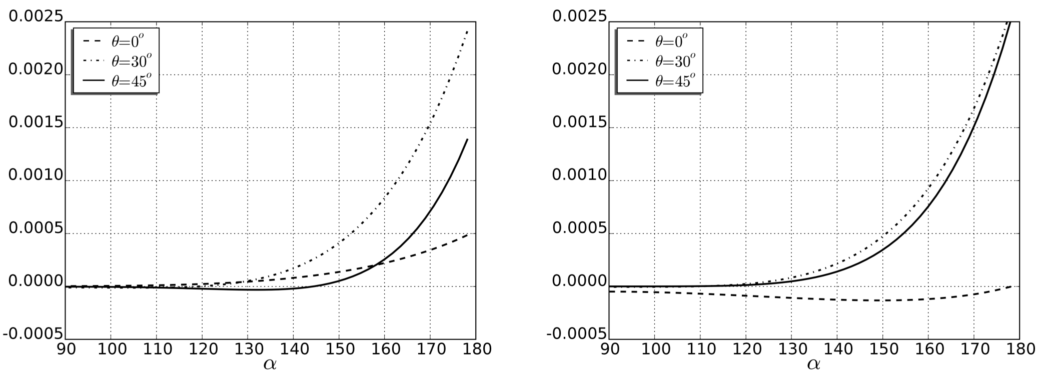

Figure 2: Phase error versus element distortion for eighth-degree spectral elements with Chebyshev (left) and Legendre (right) collocation points. The number of grid points per wavelength is five.

REFERENCES

[1] G. Cohen. *Higher-order numerical methods for transient wave equations*, Springer, 2002.

[2] G. Seriani and S. Oliveira. "Optimum blended spectral element operators for forward modeling". *Geophysics*, Vol. **72**, SM95–SM106, 2007.