8th. World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2008)
June 30 –July 5, 2008
Venice, Italy

Multi-Step Preconditioner for Saddle-Point Problems

*** Aleksandar Jemcov[1] and Joseph P. Maruszewski[2]**

[1] ANSYS/Fluent Inc.
10 Cavendish Court, Lebanon
NH, 03766, USA
aj@fluent.com
www.fluent.com

[2] ANSYS/Fluent Inc.
1007 Church Street, Suite 250, Evanston
IL, 60201, USA
jpm@fluent.com
www.fluent.com

Key Words: *Saddle-Point Problem, Preconditioner, Schur Complement, Pseudo-Inverse.*

ABSTRACT

Many computational problems require the solution of saddle point systems. Examples include the solution of the Navier-Stokes equations for incompressible flow, contact problems in solid mechanics, optimization problems, and domain decomposition methods. The most general form of this problem is

$$\mathcal{A}x = \begin{bmatrix} F & G \\ D & K \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = b \tag{1}$$

Usually the matrix K is zero but it may be nonzero due to the presence of stabilization terms when solving the Navier-Stokes equations. Also it sometimes occurs that $D = G^T$, but we choose to consider the most general case here. The matrix $\mathcal{A}$ is generally indefinite (i.e. it has eigenvalues with both positive and negative real parts).

The system (1) can be solved by Krylov subspace methods such as CG, BCGStab, TFQMR, or GMRES [1]. Since these systems are indefinite and typically very large, it is necessary to use preconditioning in order to obtain acceptable solution times. Here, we describe the new multi-step preconditioner for saddle-point problems that is based on LU factorization of the approximation of the block matrix in Eq. (1)

$$\begin{bmatrix} \tilde{F} & G \\ D & 0 \end{bmatrix} = \begin{bmatrix} \tilde{F} & 0 \\ D & \tilde{S} \end{bmatrix} \begin{bmatrix} I & \tilde{F}^{-1}G \\ 0 & I \end{bmatrix}. \tag{2}$$

Here $\tilde{S}$ is approximate Schur complement matrix $\tilde{S} = K - D\tilde{F}^{-1}G$. LU factorization in Eq. (2) is used in an iterative algorithm, thus resulting in multi-step saddle point preconditioner

$$\begin{bmatrix} \tilde{F} & 0 \\ D & \tilde{S} \end{bmatrix} \begin{bmatrix} I & \tilde{F}^{-1}G \\ 0 & I \end{bmatrix} \begin{bmatrix} u_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} - \begin{bmatrix} \left(F - \tilde{F}\right) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix}. \tag{3}$$

Performance of the saddle point preconditioner given by Eq. (3) depends on the choice of the approximate inverse used in the solution of the Schur complement problem. Here we construct pseudo-inverse of the Schur complement matrix by minimizing the Frobenius norm of the the matrix

$$\Lambda = I - \tilde{F}^{-1}F. \tag{4}$$

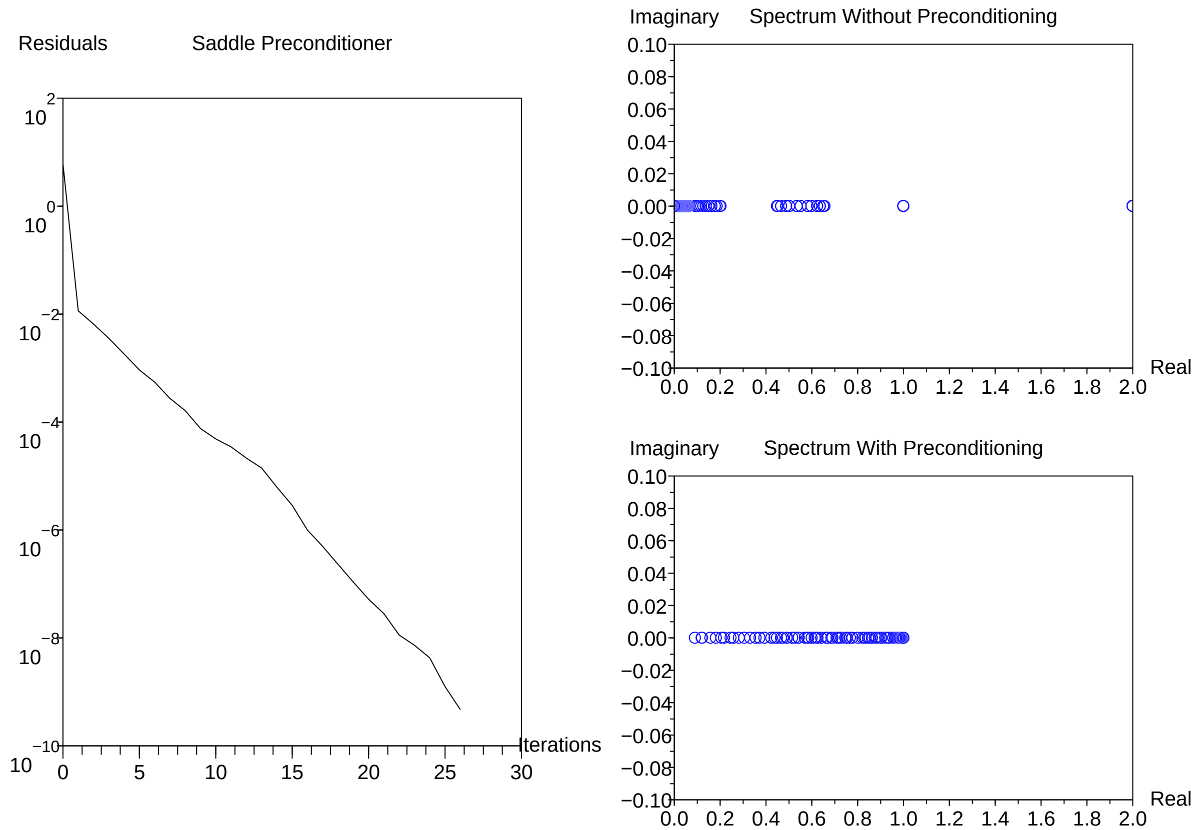

Figure 1: Performance of the saddle-point preconditioner for a driven cavity.

Here F is the first matrix on the main diagonal of the block matrix in Eq. (1) and $\tilde{F}^{-1}$ is its approximate inverse.

Application of this preconditioner is demonstrated in the case of incompressible flow problems discretized by finite element method. The performance of the new preconditioner is illustrated in Fig. (1) where an excellent convergence rate of the solver together with improved preconditioned spectrum are observed. Additional examples including large 3D problems and applications of the preconditioner to the domain decomposition simulations show that the new saddle preconditioner has favorable properties. Additional considerations regarding the implementation of the preconditioning algorithm and optimality of the pseudo-inverse are also included. Finally, the connection of the preconditioner and existing iterative methods used in computational fluid dynamics is highlighted and advantages of the proposed algorithm over the traditional methods are emphasized.

REFERENCES

[1] Y. Saad. “Iterative Methods for Sparse Linear Systems”. *PWS, Boston*,1996.