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The Convex Source Support and Its Application to Electric Impedance Tomography

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ABSTRACT

In this talk we extend the concept of the convex scattering support to the case of electrostatics in a bounded domain. Furthermore, we apply this approach to devise a new reconstruction algorithm in electric impedance tomography. The convex scattering support was developed by Stephen Kusiak and John Sylvester, cf. [3],[4], and is meant to be the smallest convex set that contains a scattering source compatible with some given data of the scattered wave.

To transfer this concept to the case of electrostatics, we consider the Poisson equation in a bounded domain $D \subset \mathbb{R}^2$ with a natural boundary condition, i.e.,

$$\Delta u = F \quad \text{in } D, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D, \qquad \int_{\partial D} u \, ds = 0, \tag{1}$$

where ν denotes the exterior unit normal of ∂D, and F a distributional source with vanishing mean and compact support in D. As our interest lies in the inverse problem, we want to gather all information about F that can be obtained from knowing the Dirchlet potential $g = u|_{\partial D}$. Since neither the source F nor even its support can be uniquely reconstructed thereby, we ask for the *smallest closed set* that carries a distributional source F which is compatible with the given data g. Therefore, we introduce the *convex source support* $\mathcal{C}g$ as the smallest *convex* set of this type. In particularly, the convex source support is a subset of the convex hull of the support of the true source F in (1).

If D is the unit disk then there is a straightforward criterion to decide whether the convex source support of g is a subset of a given disk $B \subset D$. Namely, we observe that $\mathcal{C}g$ is a subset of a concentric disk inside D with radius R, if and only if the Fourier coefficients $(\alpha_j)_{j\in\mathbb{Z}}$ of g decay sufficiently rapidly so that the series

$$\sum_{j=-\infty}^{\infty} \frac{|\alpha_j|^2}{(R+\varepsilon)^{2|j|}} < \infty$$

converges for every $\varepsilon > 0$. For a nonconcentric disk $B \subset D$ an analog criterion is obtained by investigating the potential $g \circ \Phi^{-1}$ for an adequate Moebius transform Φ that maps B onto a concentric disk and leaves D invariant.

The concept of the convex source support can now be easily adopted to solve the following variant of the inverse conductivity problem: We want to reconstruct the inclusions in an otherwise homogeneous tissue, i.e., the set Ω where the electric conductivity σ differs from one. The boundary value problem to be considered is thus the following one,

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } D, \qquad \frac{\partial u}{\partial \nu} = f \quad \text{on } \partial D, \qquad \int_{\partial D} u \, ds = 0, \tag{2}$$

where f is the boundary current (with vanishing mean) inserted into D. We want to reconstruct Ω from the boundary measurement of the electrostatic potential u, i.e., we use only one Dirichlet/Neumann pair of boundary data for problem (2). Generating the corresponding harmonic reference potential u_0 — i.e., the solution of (2) with σ replaced by one — we can reduce this problem to finding the convex source support for the Poisson equation (1), where $F = \nabla \cdot (1-\sigma)\nabla u$. This source F is always supported within the support of $1-\sigma$, and the reconstructed convex source support of the Dirichlet potential $(u-u_0)|_{\partial D}$ is hence a subset of the convex hull of the inclusion. We can now apply the above criterion to check whether the convex source support is a subset of an arbitrary disk B. In case that B does not lie within D, we harmonically continue the Dirichlet potential g onto the boundary of a sufficiently large disk containing B. With a similar Moebius transform as before the above criterion is then applicable to B. Finally, we intersect all disks which contain the convex source support of g and arrive at a reconstruction of the inclusion Ω.

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