8th. World Congress on Computational Mechanics (WCCM8)
5th. European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2008)
June 30 – July 5, 2008
Venice, Italy

FINITE ELEMENT PATTERNS TO SATISFY THE CONSISTENCY CONDITION

Wojciech Gilewski[1]

[1] Warsaw University of Technology, Faculty of Civil Engineering
Al. Armii Ludowej 16, 00-637 Warsaw, Poland
w.gilewski@il.pw.edu.pl, http://www.il.pw.edu.pl

Key Words: *Finite Element Pattern, Finite Element Template, Consistency.*

ABSTRACT

Beams, arches, plates and shells are widely considered in engineering applications. However the corresponding discretization procedures are not yet sufficiently reliable. It is difficult to obtain an element that is optimal. In a formulation we should aim to satisfy: ellipticity, consistency and the inf-sup condition. *Ellipticity* ensures that the finite element model is solvable and physically means there are no spurious zero energy modes. This condition can be verified by studying the zero eigenvalues and corresponding eigenvectors of the stiffness matrix of a single unsupported finite element. *Consistency* is related to the convergence. The finite element solution must converge to the solution of a mathematical problem when the element size h is close to zero. The bilinear forms used in the finite element discretization must approach the exact bilinear forms of the mathematical model as h approaches zero (Gilewski [3-5]). The *inf-sup* condition ensures optimal convergence in bending-dominated problems and is not a subject of this paper. One of the interesting concepts for finite element formulations are templates proposed by Felippa [1,2]. A finite element template is an algebraic form for element matrices, which contains free parameters. Setting those parameters to specific values produces element instances.

The objective of this paper is to propose a new, general method for development of finite element stiffness matrices that satisfies the consistency condition, in the sense described above. The matrices are named "patterns" to make a difference with the templates used by Felippa [1,2]. The proposed method is based on the energy-difference criterion of evaluation of finite elements, described in details for 1D and 2D problems in [3-5]. For certain problem, for which the strain energy density $\widetilde{E}_s$ in differential form is defined, the element stiffness matrix $\mathbf{K}$ is defined with as many independent parameters as possible, with the density of strain energy $\widetilde{E}_s^{FE} = \mathbf{q}^T \mathbf{K} \mathbf{q} / 2 / A$, where A is the element length (for 1D), element area (for 2D) or element volume (3D). Each parameter of the nodal displacement vector $\mathbf{q}$ can be expressed (with the use of Taylor series expansion) by the average displacement and its derivatives in the middle point of the element. The difference operators are to be compared with the differential operators of the strain energy. In the limit case $h \to 0$ the following relation should be valid

(consistency condition): $\lim_{h \to 0} \widetilde{E}_s^{FE} = \widetilde{E}_s$. This relation gives a number of equations to be satisfied by the defined finite element parameters. Some of the parameters remain free. The analysis of higher order terms of the element energy gives some additional conditions that can be satisfied to avoid shear or membrane locking of the element.

Example - Timoshenko Beam Pattern.

Element data: length $2a$, 2 nodes, 4 natural d.o.f., EJ - bending rigidity, H -shear rigidity, c_1, c_2, c_3 - free parameters. Strain energy density: $2\widetilde{E}_P^{TB} = EJ\left(\frac{d\phi}{dx}\right)^2 + H\left(\frac{dw}{dx} - \phi\right)$.

$$\mathbf{K}_P^{TB} = \frac{EJ}{2a}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} + \frac{H}{2a}\begin{bmatrix} 1 & 2a(1-2c_2) & -1 & 4c_2 a \\ 2a(1-2c_2) & 4a^2(1-2c_1-c_3) & -2a(1-2c_2) & 4a^2 c_1 \\ -1 & -2a(1-2c_2) & 1 & -4ac_2 \\ 4c_2 a & 4a^2 c_1 & -4ac_2 & 4a^2 c_3 \end{bmatrix}$$

The above pattern is similar, but not exactly the same, to the template proposed by Felippa [1,2]. The number of free parameters is 3 in both formulations. If we put $c_1 = \frac{1}{6}$, $c_2 = \frac{1}{4}$, $c_3 = \frac{1}{3}$ the stiffness matrix is received for the element with linear shape functions and full integration. For $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{4}$, $c_3 = \frac{1}{4}$ we have the element with uniformly reduced integration. The analysis of higher order terms gives the information that there are no shear locking if the free parameter $c_1 = \frac{1}{4}$.

Other FE Patterns will be presented during the conference: plane truss pattern (2 d.o.f.) with zero free parameters; Bernoulli beam pattern (4 d.o.f.) with one free parameter; Timoshenko circular arch pattern (6 d.o.f.) with 12 free parameters; rectangular Mindlin plate pattern (12 d.o.f.) with 54 free parameters. Free parameters will be discussed.

To conclude:

- The method to create the finite element pattern matrices is proposed to satisfy the consistency condition of the FE formulation.
- The patterns contain, so called, free parameters. The number of free parameters depends on the problem.
- It is possible to define the values of these parameters to avoid shear or membrane locking of the finite element.
- For given free parameters of the pattern the ellipticity condition should be checked.

REFERENCES

[1] Felippa C.A., "A template tutorial I: panels, families, clones, winners and losers", Report No. CU-CAS-03-03, University of Colorado, 2003.

[2] Felippa C.A., "The amusing history of shear flexible beam elements", Report No. CU-CAS-05-01, University of Colorado, 2005.

[3] Gilewski W., Correctness of plate bending finite element with physical shape functions, *Finite Element News*, 3, pp.29-34, (1993).

[4] Gilewski W., Evaluation of finite elements. *16th Int.Conf. on Computer Methods in Mechanics CMM-2005,* Częstochowa 2005.

[5] Gilewski W., *On the Criteria for Evaluation of Finite Elements – From Timoshenko Beam to Hencky-Bolle Plate* (in Polish), OW Politechniki Warszawskiej, Warsaw 2005.