8th. World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineeering (ECCOMAS 2008)
June 30 –July 5, 2008
Venice, Italy

ABSTRACT TITLE
High order well balanced schemes for systems of balance laws

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Key Words: *Well balanced schemes, finite volume schemes, high order, shallow water.*

ABSTRACT

Several mathematical models of great relevance in the applications are described as systems of balance laws: the time derivative of a density plus the gradient of a flux equals the source term. If the flux is a non linear function of the density, then jump discontinuities may form in finite time. Shock capturing finite volume and finite difference schemes have reached a great sophistication in the numerical treatment of systems of conservation laws, delivering highly accurate solutions in smooth regions, and sharp resolution near discontinuities.

If a source term is present, a challenging problems appears when looking for a solution which is a small perturbation of a stationary one: in the latter, flux gradient and source perfectly balance, and the solution is time independent. If such equilibrium is not maintained at a discrete level, then the discretization error may strongly pullute the numerical solution producing undesired spurious effects. This problem can be avoided by the so called *well balanced schemes*, which maintain, at a discrete level, one or all equilibria of the original system.

¿From the original work of Bermudez and Vaszques [1], a great progress has been made in the development of well balanced shock capturing schemes. In the application to shallow water, second order schemes have been constructed that preserve static equilibria. The schemes have usually been constructed for upwind or central schemes on a non staggered grids. Second order well balanced central schemes for shallow water equation can be constructed, that preserve static solution [2] as well as stationary solutions with non zero water flow [3].

For many years the second order barrier seemed a limit in the accuracy of such schemes. Only recently groups of authors have proposed several alternative ways for constructing high order well-balanced schemes (see, for example [4,5,6]).

In this paper we propose a new family of high-order well-balanced schemes for the numerical solution of systems of balance laws.

The schemes make use of two sets of variables, conservative variables u and equilibrium ones v. The equilibrium variables are defined in such a way that they are constant at equilibrium. A suitable mapping is defined between the two sets of variables. We assume that such a mapping is always one-to-one.

In the case of 1D shallow water equations, for example, conservative variables are $u = (h, q)$, while equilibrium variables are $v = (q, \eta)$, where $q = hw$ is the water flux, w the velocity, and $\eta = w^2/2 + g(h + B)$ is the energy density (which plays the role of mathematical entropy), and $B = B(x)$ is the bottom topography.

High order resolution is obtained by using high order WENO reconstruction, and a suitable high order Runge-Kutta integrator in time. The idea of the scheme is to discretize the equation in the conservative variables, while the equilibrium variables will be used in the reconstruction. The high order WENO reconstruction on the equilibrium variables is constructed in such a way to guarantee conservation.

Staggered and non staggered version of the schemes will be presented. The schemes will be able to preserve static and non static equilibria. We remark that this is the first high order scheme for central schemes on staggered grids.

The schemes can be applied to several systems in which it is possible to identify equilibrium variables. Applications will be performed to shallow water equations, nozzle flow, and gas in stratified atmosphere. The results confirm the high resolution and well-balanced properties of the schemes.

REFERENCES

[1] A. Bermudez, M.E. Vazquez. “Upwind methods for hyperbolic conservation laws with source terms”. *Comput. Fluids*, Vol. **23**, 1049–197129, 1994.

[2] G.Russo. *Central schemes for balance laws*, in Hyperbolic problems: theory, numerics, applications, Vol. I, II (Magdeburg, 2000), 821–829, Internat. Ser. Numer. Math., 140, 141, Birkhuser, Basel, 2001.

[3] G.Russo. “Central schemes for conservation laws with application to shallow water equations”, in *Trends and applications of mathematics to mechanics: STAMM 2002*, S. Rionero and G. Romano (editors), pp.225–246, Springer-Verlag Italia SRL, 2005.

[4] Sebastian Noelle, Normann Pankratz, Gabriella Puppo, Jostein R. Natvig. “Well-balanced finite volume schemes of arbitrary order of accuracy for shallow water flows”. *Journal of Computational Physics*, Vol. **213**, 474–499, 2006.

[5] Sebastian Noelle, Yulong Xing, Chi-Wang Shu. “High-order well-balanced finite volume WENO schemes for shallow water equation with moving water”. *Journal of Computational Physics*, Vol. **226**, 29–58, 2007.

[6] José M. Gallardo, Carlos Parés, Manuel Castro. “On a well-balanced high-order finite volume scheme for shallow water equations with topography and dry areas”. *Journal of Computational Physics*, Vol. **227**, 574-601, 2007.