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RBF AS MFS APPROXIMATIONS IN HIGHER DIMENSION

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ABSTRACT

Some of the most common scalar PDEs have fundamental solutions that have a radial feature. Therefore, the method of fundamental solutions (MFS) is usually associated to radial basis functions (RBF) as a particular case, when the basis functions have radial property. Here we will present a missing counterpart – some of the most common RBF approximation methods are just a particular case of the MFS boundary approximation when applied in a higher dimension. This idea was presented in [1]. In this presentation we will consider the relation between MFS boundary interpolation in dimension $d+1$ and RBF domain interpolation in dimension d, for the most commonly used RBF basis functions.

The notation is the following: capital bold letters, like $\mathbf{X} = (x_1, x_2, x_3)$ denote a point in 3D that we will associate to a simple bold letter $\mathbf{x} = (x_1, x_2) \longmapsto (x_1, x_2, 0)$ representing a 2D point, that is considered to be in the plane $x_3 = 0$. Likewise, a function defined in 3D variables will denoted by capital letter like $F(\mathbf{X})$ and its restriction to the plane $x_3 = 0$, will be denoted simply by $f(\mathbf{x})$, being $f(x_1, x_2) = F(x_1, x_2, 0)$.

RBF-MQ in 2D as the MFS for the 3D bilaplacian. Consider a function f, defined in a bounded set $\omega \subset \mathbb{R}^2$, the standard 2D RBF-MQ collocation approach with multiquadrics uses parameters c_j

$$f_N(\mathbf{x}) = \sum_{j=1}^{N} a_j \sqrt{\left|\mathbf{x} - \mathbf{y}^{(j)}\right|^2 + c_j^2}$$

or more precisely, using the notation $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y}_j = (y_1^{(j)}, y_2^{(j)})$,

$$f_N(x_1, x_2) = \sum_{j=1}^{N} a_j \sqrt{(x_1 - y_1^{(j)})^2 + (x_2 - y_2^{(j)})^2 + c_j^2}.$$

Note that being Δ the Laplace differential operator in dimension d, we have for $0 \neq \mathbf{X} \in \mathbb{R}^d$,

$$\Delta(|\mathbf{X}|^p) = p(d+p-2)\,|\mathbf{X}|^{p-2}\,.$$

In particular, in 3D, it is well known that $-\Delta(|\mathbf{X}|^{-1}) = 4\pi\delta$ (being δ the Dirac delta), and therefore a fundamental solution of the bilaplacian in 3D is given by

$$\Phi_{\Delta\Delta}(\mathbf{X}) = \frac{-1}{8\pi}\,|\mathbf{X}|$$

because $\Delta^2(|\mathbf{X}|) = \Delta(\Delta(|\mathbf{X}|)) = \Delta(2\,|\mathbf{X}|^{-1}) = -8\pi\delta$.
Now consider the following MFS bilaplacian 3D framework for the boundary approximation:
(i) boundary points $\mathbf{X}^{(i)} = (\mathbf{x}^{(i)}, 0) = (x_1^{(i)}, x_2^{(i)}, 0) \in \omega \times \{0\} = \Gamma_\omega \subset \mathbb{R}^3$
(ii) source points $\mathbf{Y}^{(j)} = (y_1^{(j)}, y_2^{(j)}, c_j) \in \hat{\Gamma}_\omega \subset \mathbb{R}^3$
Considering a discrete set of 3D source points $\mathbf{Y}^{(j)} = (x_1^{(j)}, x_2^{(j)}, c_j)$, and fitting the coefficients a_j by collocation, we easily get the MFS form in 3D of the RBF-MQ approximation in 2D

$$f_N(\mathbf{x}) = -8\pi \sum_{j=1}^{N} a_j \Phi_{\Delta\Delta}(\mathbf{X} - \mathbf{Y}^{(j)}).$$

This immediately relates the RBF-MQ approximation in 2D domain ω with constant parameter $c_j = c$ to the MFS boundary approximation for the 3D bilaplacian using a parallel artificial boundary $\hat{\Gamma}_\omega = \omega \times \{c\}$. This is just a special case of the MFS framework where the true boundary is $\Gamma \supset \Gamma_\omega$ and the artificial boundary is $\hat{\Gamma} \supset \hat{\Gamma}_\omega$, that can be inserted in the usual context of MFS approximation, where density is proven, ie. $\Gamma = \partial\Omega$ and $\hat{\Gamma} = \partial\hat{\Omega}$ with $\bar{\Omega} \subset \hat{\Omega}$ open bounded sets in 3D.

RBF-IMQ in 2D as the MFS for the 3D laplacian. Using inverse multiquadrics (IMQ), the 2D RBF-IMQ collocation approach leads to

$$f_N(\mathbf{x}) = \sum_{j=1}^{N} b_j / \sqrt{\left|\mathbf{x} - \mathbf{y}^{(j)}\right|^2 + c_j^2}$$

Considering the fundamental solution of the laplacian in 3D, given by $\Phi_\Delta(\mathbf{X}) = \dfrac{-1}{4\pi\,|\mathbf{X}|}$,
with the same MFS 3D framework (i) + (ii), and considering the same source points $\mathbf{Y}^{(j)}$, we get the MFS form of the RBF-IMQ approximation

$$f_N(\mathbf{x}) = -4\pi \sum_{j=1}^{N} a_j \Phi_\Delta(\mathbf{X} - \mathbf{Y}^{(j)}).$$

Remark: As stated in [1], other connections can be established. For instance, the extra dimension could be time or frequency. In time, Gaussian RBF are related to the MFS for the heat equation. In frequency, new Helmholtz RBF can be derived from the MFS (cf.[3]), and can be related to the PDE Kansa type RBF method (cf.[2]).

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