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Reliability Analysis with Stochastic Finite Elements and Wavelet Approximations

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ABSTRACT

Polynomial chaos expansions of response quantities have been widely utilized in Computational Stochastic Mechanics and are well documented [1]. Introduced in conjunction with a truncated series representation of the input random field, they represent global approximations on a Hilbert space of functions of (usually standard Gaussian) random variables. In the context of reliability estimation, stochastic response surfaces obtained by local chaos expansions have been developed [2]. They depend on three parameters: the truncation of the series representation of the input random field, a mesh parameter in the physical space and a mesh parameter in the random parameter space. Thus, a stochastic response surface can be viewed as a member of a three parameter family of metamodels.

However, the discretization of the random field may enforce a rather fine FE mesh and thus increase the computational effort. In this paper, a wavelet expansion of the eigenfunctions in the series representation of the random field is introduced in order to control the spatial resolution of the eigenfunctions. Finer scales are taken into account by projection operators. The truncation level of the wavelet expansion has been identified as an additional parameter of the metamodel.

Let D be a convex bounded open set in $\mathbb{R}^n$ and $(\Omega, \mathcal{F}, P)$ a complete probability space, where Ω is the set of outcomes, $\mathcal{F}$ the σ-field of events and $P : \mathcal{F} \rightarrow [0 : 1]$ a probability measure. Consider the following model problem on $\bar{D} \times \Omega$: find $u : \bar{D} \times \Omega \rightarrow \mathbb{R}$, such that almost surely:

$$-\nabla \cdot \alpha(x,\omega)\nabla u(x,\omega) = f(x) \text{ on } D,\ u(x,\omega) = 0 \text{ on } \partial D \quad (1)$$

where $L(x)$ is a deterministic differential operator and $\alpha : D \times \Omega \rightarrow \mathbb{R}$ is a random field.

The random field $\alpha(x,\omega)$ is replaced by a truncated Karhunen-Loève expansion:

$$\alpha(x,\omega) \approx E[\alpha](x) + \sum_{i=1}^{N} \sqrt{\lambda_i}\xi_i(\omega)\phi_i(x), \quad (2)$$

where $\xi_i(\omega)$ are zero mean, unit variance and uncorrelated random variables. The constants λ_i and the orthonormal deterministic functions $\phi_i(x)$ are related to the eigenvalue problem for the covariance kernel $C(x, y) = \int_\Omega (\alpha(x, \omega) - E[\alpha](x))(\alpha(y, \omega) - E[\alpha](y))dP(\omega)$:

$$\int_D C(x, y)\phi_i(y)dy = \lambda_i \phi_i(x). \tag{3}$$

The random variables can be expressed by the reproducing kernel representation

$$\xi_i(\omega) = \frac{1}{\lambda_i} \int_D (\alpha(x, \omega) - E[\alpha](x))\phi_i(x)dx. \tag{4}$$

The eigenfunctions are further decomposed on a Haar wavelet basis. These approximations have been introduced by several authors (see e.g. [3]) for simulating random fields. Here, the decomposition is introduced in order to obtain a coarse approximation (by taking scales until a certain level m into account) and a fine approximation (scales from $m + 1$ to n).

For the numerical solution of the stochastic boundary value problem a variational formulation on the tensor product Hilbert space $H_0^1(D) \otimes L_2(\Omega)$ is introduced. Standard piecewise continuous polynomial approximations are applied in $H_0^1(D)$, and h-approximations [4] are utilized in $L_2(\Omega)$. To this end, the domain $\prod_{i=1}^N \xi_i(\Omega)$ is partitioned into non-overlapping domains. These domains serve as support of the approximation functions, which are polynomials in $\xi(\omega),\ i = 1, ..., N$, on one domain and vanish elsewhere.

Introducing the approximations into the variational formulation of the boundary value problem together with a Galerkin technique leads to the following algebraic problem for the determination of the coarse scale solution at the nodes of the finite element mesh: $K_c u_c = F$. In a second step, a correction Δu to u_c is computed from a residuum r that is obtained is by projecting of u_c on a fine mesh, applying the fine scale linear system matrix and interpolating on the coarse mesh: $K_c \Delta u = -r$.

Once the algebraic problems are solved and the correction to the coarse scale solution is computed, an approximation for $u(x, \omega)$ has been obtained, that can be considered as a response surface, which is, if a h- or hp-method is adopted, of local character. The approximation quality depends on four parameters: the FE mesh, the truncation of the Karhunen-Loève expansion (N), the stochastic grid (size and location of the subdomains) and the wavelet scales. These parameters can be adapted with respect to the point of most probable failure.

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