8th. World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineeering (ECCOMAS 2008)
June 30 –July 5, 2008
Venice, Italy

Convergence of AVIs for the linear elastodynamics of simple bodies

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Key Words: *Asynchronous variational integrators, discrete dynamics, convergence.*

ABSTRACT

Asynchronous variational integrators (AVIs) are a class of algorithms that allow to discretize continuous models of evolutionary phenomena and preserve their simplectic structure, when it is present. When the phenomenon under analysis is composed by subsystems, AVIs allow one to attribute to each subsystem different time discretizations in a way useful to assure that the energy is "nearly" preserved along computations in the conservative case (see [3],[4]).

A natural playground to test AVIs is the linear elastodynamic of simple bodies. Let us consider a simple elastic body undergoing infinitesimal deformations that occupies a regular region $\mathcal{B}$ of the three-dimensional ambient space. For $x \in \mathcal{B}$, $t \in [t_0, t_f]$ the differentiable map $u(x,t)$, with values in $\mathbf{R}^3$, represents the *displacement field*. The map $\varepsilon := sym\nabla u(x,t)$ associates the standard *measure* ε *of infinitesimal deformations*. In linear elastic constitutive setting and infinitesimal deformation regime, the dynamic of a simple body is governed by the action functional

$$\mathcal{A}(\mathcal{B};u) := \int_{t_0}^{t_f} \left(\int_{\mathcal{B}} \left(\frac{1}{2}\rho \left|\dot{u}\right|^2 - \frac{1}{2}\left(\mathbf{C}\varepsilon\right)\cdot\varepsilon + \mathbf{b}\cdot u \right) dx + \int_{\partial\mathcal{B}_{\mathtt{t}}} \mathtt{t}\cdot u \, d\mathcal{H}^2 \right) dt, \tag{1}$$

where ρ is the density of mass, $\mathbf{C}$ the standard elastic constitutive tensor, $\mathbf{b} \in L^2\left(\mathcal{B}, \mathbf{R}^3\right)$ and $\mathtt{t} \in L^2\left(\partial\mathcal{B}, \mathbf{R}^3\right)$ bulk and surface conservative forces, respectively. The *traction* $\mathtt{t}$ is applied over a closed subset $\partial\mathcal{B}_t$ of $\partial\mathcal{B}$. Initial conditions are given by regular fields $u_0(x,t_0)$ and $\dot{u}_0(x,t_0)$.

For constructing computational schemes based on the direct discretization of the action both in space and time, one selects a tessellation $\mathcal{T}$ of $\mathcal{B}$ of finite elements and consider functions $u(x,t) = \sum_{a\in\mathcal{T}} \mathcal{N}_a(x)u_a(t)$, where $\mathcal{N}_a$ is shape function corresponding to node a ($\mathcal{N}_a$ is assumed to be a piecewise linear polynomial). The sole space discretization leads to an action $\mathcal{A}_{\mathcal{T}}$ defined by

$$\begin{aligned}\mathcal{A}_{\mathcal{T}}(u) := &\sum_{K\in\mathcal{T}} \int_{t_0}^{t_f} \left(\sum_{a\in K} \frac{m_{K,a}}{2} \left|\dot{u}_a(t)\right|^2 - \int_K \frac{1}{2}\left(\mathbf{C}\nabla u_K(x,t)\right)\cdot\nabla u_K(x,t)\, dx \right) dt \\ &+ \sum_{K\in\mathcal{T}} \int_{t_0}^{t_f} \left(\int_K \mathbf{b}(x)\cdot u_K(x,t)\, dx + \int_{\partial K\cap\partial\mathcal{B}_t} \mathtt{t}(x)\cdot u_K(x,t)\, d\mathcal{H}^2 \right) dt, \end{aligned} \tag{2}$$

where $m_{K,a}$ is the *elemental nodal mass* of node a associated to K, $u_K(x,t)$ is the restriction of $u(x,t)$ to K and $u_K(t)$ is the vector of nodal displacements relative to element K in $\mathcal{T}$.

To select a time discretization we endow each element K of $\mathcal{T}$ with a discrete time set $\Theta_K = \left\{t_0 = t_K^1 < ... < t_K^{N_K-1} < t_K^{N_K} = t_f\right\}$. We indicate by Θ the entire time set for $[t_0, t_f]$ defined by $\Theta := \cup_{K\in\mathcal{T}}\Theta_K$, and as a measure of asynchronicity of Θ we consider the ratio

$$M_\Theta := \frac{\max_K \max_j (t_K^{j+1} - t_K^j)}{\min_K \min_j (t_K^{j+1} - t_K^j)}. \tag{3}$$

Then a discrete action sum

$$\mathcal{A}_{\mathcal{T},\Theta}(u) := \sum_{K\in\mathcal{T}} \sum_{t_K^j \in \Theta_K} \mathcal{A}^j\left(K; u_K\right) \tag{4}$$

arises. Here u_K indicates the nodal displacements in the element K, and

$$\mathcal{A}^j\left(K; u_K\right) := \sum_{a\in K} \sum_{\left\{i | t_a^i \in [t_K^j, t_K^{j+1})\right\}} \frac{1}{2} m_{K,a} \left(t_a^{i+1} - t_a^i\right) \left|\dot{u}_a\left(t_a^i\right)\right|^2 - (t_K^{j+1} - t_K^j) V_K(u_K(t_K^{j+1})).$$

AVIs are then a recursive rule that allows to calculate discrete trajectories from the (discrete) Euler-Lagrange equation of $\mathcal{A}_{\mathcal{T},\Theta}$ with given initial data. Their convergence has been proved variously: A repeated use of Gronwall inequality on the discrete Euler-Lagrange equations is the basic ingredient of the technique introduced in [4] for potentials with uniformly bounded second derivative. Γ-convergence methods have been used in [5] to study the case of the elementary (zero-dimensional) oscillator. In [1] we enlarge the stage and adapt the techniques developed in [5] to analyze the convergence of AVIs for the linear elastodynamic of a three-dimensional body. Our main result is the theorem below.

Theorem 1 (Convergence in time [1]) *Let $(\Theta_h)_{h\in\mathbf{N}}$ be a sequence of entire time sets for $[t_0, t_f]$ such that $\max_{\Theta_h}(t_{i+1} - t_i)$ is infinitesimal as $h \to +\infty$. Let also $u_h(t_0)$ and $\dot{u}_h(t_0)$ be initial conditions satisfying*

$$\sup_h \left(M_{\Theta_h} + |u_h(t_0)| + |\dot{u}_h(t_0)|\right) < +\infty.$$

Then any sequence $(u_h)_{h\in\mathbf{N}}$, with u_h a stationary point of the discrete action $\mathcal{A}_{\mathcal{T},\Theta_h}$, is pre-compact in the weak-$*$ $W^{1,\infty}\left((t_0, t_f), \mathbf{R}^N\right)$ topology and all its cluster points are stationary points for the action $\mathcal{A}_\mathcal{T}$ (here N is the total number of nodal degrees of freedom in the tessellation $\mathcal{T}$).

A different approach has been developed in [2], where the dynamic of complex bodies has been analyzed. Under the same assumptions of Theorem 1 it is possible, in fact, to establish *BV* estimates for the velocities of discrete stationary points directly from the discrete Euler-Lagrange equations. In turn these estimates permit us to pass to the limit directly into the discrete equations.

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