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# An iterated upwind method for incompressible and compressible fluids

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## ABSTRACT

There is a continuing interest in developing general template algorithms capable of solving both incompressible and compressible flows at all speed regimes. Contributions of Professor O.C. Zienkiewicz towards this goal started with a formulation of fractional finite element method applicable to incompressible and low-speed compressible flows [7]. This work was followed by a series of new developments e.g. [6] resulting in a current range of Characteristic Based Schemes, which have been applied to a variety of applications involving both incompressible and compressible fluids.

In this paper we document developments based on another flexible template algorithm for fluid flows, offered by a high resolution "nonoscillatory forward-in-time" (NFT) schemes. NFT schemes have been put forward in the early nineties in the context of geophysical flows [2,3]. NFT labels a class of second-order-accurate, either semi-Lagrangian (trajectory wise) or Eulerian (finite-volume wise), algorithms that rely on strengths of two-time-level nonlinear advection techniques that suppress/reduce/control numerical oscillations characteristic of higher-order linear schemes.

The underlying theory derives from the truncation-error analysis of upwind approximations for an archetype inhomogeneous PDE for fluids

$$\frac{\partial \phi}{\partial t} + \nabla \bullet (\mathbf{V}\phi) = R \; , \tag{1}$$

in abstraction from any particular assumptions on the nature of physical forcings $R$, and their relation to dependent-variables densities $\phi$, but driven solely by the requirement of the second-order accuracy for arbitrary flows $\mathbf{V}(\mathbf{x}, t)$. One resulting template algorithm for (1) takes a simple compact form

$$\phi_i^{n+1} = \mathcal{A}_i\big(\phi^n + 0.5\delta t R,\ \mathbf{V}^{n+1/2}\big) + 0.5\delta t R^{n+1} \; , \tag{2}$$

where $n$ and $i$ refer to the temporal and spatial position on the mesh, $\mathcal{A}$ symbolizes any two-time level nonoscillatory transport operator (viz. advection scheme) assumed second-order accurate for a homogeneous case of (1) with time-independent $\mathbf{V}$. Furthermore, $\mathbf{V}^{n+1/2}$ is a $\mathcal{O}(\delta t^2)$ estimate of $\mathbf{V}$ at

$t + 0.5\delta t$, and $R^{n+1} = R(t + \delta t) + \mathcal{O}(\delta t^3)$ can either be an explicit estimate or depend implicitly on all relevant problem variables $\phi$.

In principle, NFT solvers such as (2) should be adaptable to multi-physical applications and all speed flows. Notwithstanding their proven record for incompressible and anelastic flows using structured meshes, their extension to compressible flows was exposed only recently [5]. Herein, we complement our earlier work with a new development of NFT scheme for incompressible flows operating on unstructured meshes. We use a dual mesh, edge-based framework [4].

In NFT schemes there are two elements defining the solver: i) the choice of the transport operator $\mathcal{A}$; and ii) the approach for evaluating $R^{n+1}$ at the rhs. For the former, our method of choice is MPDATA (Multidimensional Positive Definite Advection Transport Algorithm) [1], for its genuine multidimensionality (free of splitting errors), easy accuracy-sustaining generalization to unstructured meshes, and suitability for DNS, LES, and ILE. MPDATA employs iterated upwind technologies. The evaluation of $R^{n+1}$ depends on the fluid flow regime, adopted form of the conservation laws, and a selection of dependent variables. For the implicit integration of $R^{n+1}$ required for incompressible flows we use a preconditioned non-symmetric Krylov-subspace elliptic solver.

The resulting, NFT based, unified framework is applicable to both incompressible and compressible flows including problems with complex geometries for which unstructured meshes are desirable. It provides consistent — second order accurate and stable, solutions to flows with free-stream Mach numbers ranging from zero to supersonic. Theoretical considerations are supported with numerical examples from multidisciplinary applications illustrating flexibility and robustness of the approach.

The general NFT scheme uses several non-standard concepts which will be highlighted. They include: an analytical form of the leading error applied to mesh adaptivity (either a priori or a posteriori options are possible); a low-numerical diffusion method for alleviating numerical oscillations resulting e.g. from the use of non-staggered meshes; and a naturally arising from the scheme shock wave treatment. The method has also an inherent self-regulating property suitable for Implicit Large Eddy Simulations (ILES).

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