8th. World Congress on Computational Mechanics (WCCM8)
5th European Congress on Computational Methods in Applied Sciences and Engineeering (ECCOMAS 2008)
June 30 –July 5, 2008
Venice, Italy

STRICT ERROR BOUNDS FOR LINEAR AND NONLINEAR SOLID MECHANICS PROBLEMS USING THE SUBDOMAIN-BASED FLUX-FREE METHOD

*** Régis Cottereau[1,2], Pedro Díez[2] and Antonio Huerta[2]**

[1] CIMNE
Univ. Politècnica de Catalunya
Barcelona, Spain
regis.cottereau@upc.edu

[2] LaCàN
Univ. Politècnica de Catalunya
Barcelona, Spain
{pedro.diez,antonio.huerta}@upc.edu

Key Words: *Error estimation, Error bounds, Residual-based estimators, Nonlinearity.*

ABSTRACT

In this paper, we consider the problem of the *a posteriori* error estimation associated with the computation of a finite element approximate solution $\boldsymbol{u}_H \in \boldsymbol{\mathcal{V}}_H$ of a linear or nonlinear problem in elasticity. A classical approach [1] consists in setting the error estimation problem in a residual form as

$$a_\Omega(\boldsymbol{z}, \boldsymbol{v}) = \ell(\boldsymbol{v}) - a_\Omega(\boldsymbol{u}_H, \boldsymbol{v}) =: R(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}},$$

where $a_\Omega(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v}) d\omega$, $\ell(\boldsymbol{v})$ is the forcing function, $R(\boldsymbol{v})$ is the residual, and the constitutive relation $\boldsymbol{\sigma} = \mathbb{C}(\epsilon)$ is possibly nonlinear.

Several methods [1] start from that global problem and replace it by a series of local problems, posed either on the elements of the finite element mesh, or on the so-called stars, which are the unions of the elements touching one of the N_v vertices of the finite element mesh. Most of these approaches require the computation of equilibrated fluxes at the global level, that are then used as boundary conditions on the local problems. Flux-free methods [2,3] were also developed that bypass this equilibration requirement by using the fact that the family of finite element interpolation functions is a partition of unity. We follow here more particularly the approach proposed in [3]. The local problems are then cast as: find $\boldsymbol{z}^i \in \boldsymbol{\mathcal{V}}_{\Omega_H^i}$ such that

$$a_i(\boldsymbol{z}^i, \boldsymbol{v}) = R\left(\Phi_H^i(\boldsymbol{v} - \Pi\boldsymbol{v})\right), \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}_{\Omega_H^i}, \quad 1 \le i \le N_v,$$

where $\boldsymbol{\mathcal{V}}_{\Omega_H^i}$ is the restriction of $\boldsymbol{\mathcal{V}}_H$ on Ω_H^i, and extended to Ω with zeros. The introduction of the operator Π is necessary when considering the very popular situation of linear interpolation functions in the finite element approximation, to ensure the solvability of the local problems. A pointwise projection operator was proposed in [3] for Π, but it cannot be used in the dual formulation that will be introduced further on. We therefore introduce a new operator $\Pi_\Omega : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{V}}^H$ defined by

$$\Pi_\Omega \boldsymbol{v}(\boldsymbol{x}) = \sum_{i=1}^{N_v} \Psi_H^i(\boldsymbol{x}) \int_\Omega \Phi_H^i \boldsymbol{v}\, d\Omega = \sum_{i=1}^{N_v} \Phi_H^i(\boldsymbol{x}) \int_\Omega \Psi_H^i \boldsymbol{v}\, d\Omega,$$

where the $\{\Psi_H^i\}_{1\leq i\leq N_v}$ are defined by

$$\Psi_H^i = \sum_{j=1}^{N_v} [M^{-1}]_{ij}\ \Phi_H^j(\boldsymbol{x}), \quad 1 \leq i \leq N_v$$

with $[M^{-1}]$ the inverse of the mass matrix $[M]$ defined by $[M]_{ij} = \int_\Omega \Phi_H^i \Phi_H^j\, d\Omega$, $1 \leq i, j \leq N_v$.

Following the work in [4,5], a dual formulation based on the principle of minimizing the complementary energy is introduced to obtain a guaranteed upper bound for $\boldsymbol{z}$. In practice, we introduce on each star Ω_H^i, a new variable $\boldsymbol{q}^i$, in the space of second-order tensors with coordinates in L^2, representing a statically admissible stress field over Ω_H^i, in the sense that

$$\begin{cases} \mathrm{Div}_{\boldsymbol{x}}\boldsymbol{q}^i = (\Pi_\Omega^{\Omega *} - \mathrm{Id})(\Phi_H^i \boldsymbol{f}_H) - \Pi_\Omega^{\Gamma^{\mathrm{N}} *}(\Phi_H^i \boldsymbol{g}_H) + \Pi_\Omega^{\Gamma_{\mathrm{int}} *}(\Phi_H^i \boldsymbol{j}_H) & \text{in } \Omega_H^i \\ \boldsymbol{q}^i \cdot \boldsymbol{n} = \Phi_H^i \boldsymbol{g}_H & \text{on } \Gamma^{\mathrm{N}} \cap \partial\Omega_H^i \\ [\![\boldsymbol{q}^i \cdot \boldsymbol{n}]\!] = -\Phi_H^i \boldsymbol{j}_H & \text{on } \Gamma_{\mathrm{int}} \end{cases},$$

where $\boldsymbol{f}_H$, $\boldsymbol{g}_H$ and $\boldsymbol{j}_H$ are known explicitely and depend on the data of the problem and on the approximate solution $\boldsymbol{u}_H$. The operators $\Pi_\Omega^{\Omega *}$, $\Pi_\Omega^{\Gamma^{\mathrm{N}} *}$ and $\Pi_\Omega^{\Gamma_{\mathrm{int}} *}$ all derive explicitely from the definition of Π_Ω and have their image in $\boldsymbol{\mathcal{V}}_{\Omega_H^i}$. Thanks to the definition of these operators, when all physical loads over Ω are polynomial, so are those of the local problem above. Hence the results in [4,5] hold and $\boldsymbol{q}^i$ can be explicitely constructed over each star.

For linear constitutive relations, we then have that $2\pi_c(\hat{\boldsymbol{q}}) \geq \|\boldsymbol{z}\|_\Omega^2$, where $\pi_c(\cdot)$ is the complementary energy and $\hat{\boldsymbol{q}} = \sum_{i=1}^{N_v} \boldsymbol{q}^i$. Provided that certain conditions of convexity [6] or monotonicity [7] of the potentials associated with the constitutive relation are verified, corresponding bounds for nonlinear problems can also be derived, based on that statically admissible stress field $\hat{\boldsymbol{q}}$.

Examples in 2D and 3D solid mechanics assert the presented results.

REFERENCES

[1] M. Ainsworth and J.T. Oden. *A posteriori error estimation in finite element analysis*, Wiley-Interscience, 2000.

[2] I. Babuška and W.C. Rheinboldt. "Error Estimates for Adaptive Finite Element Computations". *SIAM J. Numer. Anal.*, Vol. **15**(4), 736–755, 1978.

[3] N. Parès, P. Díez and A. Huerta. "Subdomain-based flux-free a posteriori error estimators". *Comp. Meths. Appl. Mech. Engr.*, Vol. **195**(4-6), 297–323, 2006.

[4] A.M. Sauer-Budge, J. Bonet, A. Huerta and J. Peraire. "Computing bounds for linear functionals of exact weak solutions to Poisson's equation". *SIAM J. Num. Anal.*, Vol. **42**(4), 1610–1630, 2004.

[5] N. Parès, J. Bonet, A. Huerta and J. Peraire. "The computation of bounds for linear-functional outputs of weak solutions to the two-dimensional elasticity equations". *Comp. Meths. Appl. Mech. Engr.*, Vol. **195**(4-6), 406–429, 2006.

[6] P. Ladevèze. Upper error bound on calculated outputs of interest for linear and nonlinear structural problems". *Comptes-Rendus Mécanique*, Vol. **334**, 399–407, 2006.

[7] P. Ladevèze. "Strict upper bounds on computed outputs of interest in computational structural mechanics". *Comp. Mech.*, Available online, 2007.