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NUMERICAL SIMULATION IN BIOMECHANICS A FLEXIBLE MULTISCALE APPROACH

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ABSTRACT

The numerical simulation of biomechanical problems based on patient specific models puts high demands an the efficiency and reliability of the simulation methods as well as on the representation of the geometries under consideration. We present an abstract multiscale framework for the parallel solution of non–smooth and non–convex minimization problems, as arise from the modeling of soft tissues and biomechanical processes.

Computing large deformations of elastic bodies, such as soft tissues or rubber, yields non–convex and constrained minimization problems. After discretizing this minimization problems by, e.g., finite elements, it can be written as

$$u_h \in \mathcal{B}_h \subset S_h : \qquad J(u_h) = \min! \tag{1}$$

where S_h denotes the finite element space, $J : S_h \to \mathbb{R}$ is the at least continuous differentiable elastic energy and $\mathcal{B}_h = \{v \in S_h \mid \underline{\phi} \leq v \leq \overline{\phi}\}$ is a convex and non-empty set of admissible solutions depending on $\underline{\phi}, \overline{\phi} \in S_h$, given a priori. For one-sided contact problems and Lagrangian finite elements of first order, for example, the set $\mathcal{B}_h$ of admissible displacements takes the form $\mathcal{B}_h = \{v \in S_h \mid v(p) \cdot n(p) \leq \overline{\phi}(p),\ p \in \mathcal{C}_h\}$. Here $\mathcal{C}_h$ is the set of all nodes on the potential contact boundary of the finite element mesh associated with S_h, $n(p)$ is some given normal direction at p, and $\overline{\phi}$ is the gap function measuring the distance in direction n to the obstacle.

The efficient solution of this type of minimization problem tends to be a difficult task even for convex energies J due to the presence of the constraints $\underline{\phi}, \overline{\phi}$. For non-convex energies, moreover, one additionally has to employ a globalization strategy (like, e.g., a trust–region strategy [CL94, GST06, GK08]) in order to succeed in computing at least a local minimizer. We present an abstract multiscale approach, which allows for the treatment of constrained convex minimization problems as well as non-convex minimization problems. To this end, we introduce the multilevel splitting

$$S_h = S_j \supsetneq \ldots \supsetneq S_0$$

which may be associated with finite element discretizations for different mesh-sizes $h_0 > \cdots > h_j$. In case of convex energies J, we follow the idea of monotone multigrid methods, see [KK01]: in each

of the subspaces S_i successive corrections s_i are applied creating a sequence $(u_\nu)_{\nu>0} \subset \mathcal{B}_h$ such that $J(u_0) \geq J(u_1) \geq \cdots$ and u_ν converges to the minimizer of (1). In order to guarantee the admissibility of the iterates u_ν, the coarse grid spaces S_i for $0 \leq i < j$ are equipped with non-standard and non-linear basis functions, which allow for a representation of the active set $\mathcal{A}(u_\nu) = \{p \in \mathcal{C}_h \mid v(p) \cdot n(p) = \overline{\phi}(p)\}$. This induces a solution dependent multilevel splitting $S_h = S_j \supsetneq S_{j-1}(u_\nu) \ldots \supsetneq S_0(u_\nu)$ which allows for the robust and efficient solution of convex constrained minimization problems of the form (1) with optimal complexity, cf., [KK01,K05].

These methods can be generalized to the case of frictional contact and saddle point systems as arising from the discretization of biphasic materials as, e.g., cartilage. For the case of friction, for example, the non-convex frictional energy has to be resolved only within the fine grid space $S_j = S_h$, whereas on the coarser levels $0 \leq i < j$ a constrained quadratic approximation is used for increasing the convergence speed of the multiscale method. This requires the usage of level dependent energies J_i on different levels i, which is different to the case of linear problems but reflects the non-linear and non-smooth structure of the frictional contact problem.

In case of non-convex energies J, the subspace corrections s_i have to be balanced carefully with the energy J in order to ensure the global convergence of the non-linear multiscale iteration process. Most globalization strategies require the solution of large-scale quadratic minimization problems like

$$s_\nu \in S_h : \qquad \langle s_\nu, \nabla J(u_\nu)\rangle + \frac{1}{2}\langle s_\nu, B(u_\nu)s_\nu\rangle \qquad \text{w.r.t.} \quad \|s_\nu\|_\infty \leq \Delta_\nu \text{ and } u_\nu + s_\nu \in \mathcal{B}_h \qquad (2)$$

where $B(u_\nu) \approx \nabla^2 J(u_\nu)$, and Δ_ν is the trust region radius, cf., [CL94]. Within this trust region, the quadratic model (2) is assumed to be a good approximation of the non-quadratic energy J. If $B(u_\nu)$ is a symmetric, positive definite matrix this enables us to employ a projected cg-method or monotone multigrids [KK01] to solve (2) sufficiently accurate.

However, using the decomposition $S_j = S_h \supsetneq \ldots \supsetneq S_0$, we can also follow [GK04, GST06] and propose a multilevel trust–region algorithm. It's paradigm is to use a projection of a current fine-level iterate and a particular coarse level model to compute a coarse level correction. This correction is interpolated and applied if a certain fine level energy reduction is achieved. In particular, after projecting the current iterate on level i, i.e. $u_{i-1} = P_i u_i \in S_{i-1}$, a trust–region algorithm is employed to solve

$$s_{i-1} \in S_{i-1} : J(u_{i-1} + s_{i-1}) + \langle s_{i-1}, R_i \nabla J(u_i) - \nabla J(u_{i-1})\rangle \qquad \text{w.r.t. } u_{i-1} + s_{i-1} \in \mathcal{B}_{i-1}$$

where $R_i : S_i \to S_{i-1}$ is a restriction operator and $P_i : S_i \to S_{i-1}$ the L^2 projection. Moreover, $\mathcal{B}_{i-1}$ is chosen such that $u_i + R_i^T s_{i-1} \in \mathcal{B}_i$. Hence, this approach yields a fully non–linear multigrid algorithm, which shows to be highly efficient for applications in non-linear mechanics and biomechanics..

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