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Numerical treatment of ordinary differential equations with multiple mixture of integer and fractional derivatives

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ABSTRACT

At present ordinary differential equations including multiple mixture of integer and fractional derivatives are a natural extension of integer-order differential equations. Notice that fractional calculus have been used in mechanics [2] and many other fields. However the solution of a equation strongly depends on the equation form. It should be noted that an analytical solution is extremly limited to the linear form of equations and includes special functions such as the Mittag-Leffler [4] and others. On the other hand, a numerical approach [1] is an alternate solution to analytical one.

In this study we will try to extend our previous numerical approach [3] to ordinary differential equations where multiple mixture of integer and fractional operators occur. We start with a classical ordinary differential equation defined as

$$D^2x(t) + a_1D^1x(t) + a_2x(t) = 0 \tag{1}$$

where $D^2 = \frac{d^2}{dt^2}$, $D^1 = \frac{d}{dt}$ denotes integer derivatives, a_1, a_2 are coefficients. We also assumed initial conditions as $x(0) = x_0$ and $D^1x(0) = \dot{x}_0$. Next we propose another equation in the following form

$$D^2x(t) + b_1\, {}_0D^{\alpha}x(t) = 0 \tag{2}$$

where b_1 is a coefficient which is assumed as $b_1 = a_1^{\alpha}a_2^{1-\alpha}$ and ${}_0D^{\alpha}$ denotes a fractional derivative of a real order $\alpha \in [0, 1)$. According to [4] we assume that the fractional derivative of arbitrary order α is defined as the left-side Riemman-Liouville derivative

$${}_0D^{\alpha}x\left(t\right) = \frac{1}{\Gamma\left(n-\alpha\right)}\frac{d^n}{dt^n}\int\limits_0^t \frac{x\left(\tau\right)}{\left(t-\tau\right)^{\alpha-n+1}}d\tau \tag{3}$$

where $n = [\alpha] + 1$ and $[\cdot]$ is an integer part of a real number. In Eqn. (2) we assumed the same initial conditions as for Eqn. (1). It should be noted that Eqn. (2) is so called an equation with the

memory effect. Both equations (1) and (2) are linear equations and therefore is possible to find analytical solutions. Such solutions are necessary for direct comparison with numerical results in order to validate our approach [3].

Next we propose a multi-term ordinary differential equation in the following form

$$D^2x(t) + c_1\, {}_0D^{\alpha_1}x(t) + c_2\, {}_0D^{\alpha_2}x(t) + c_3x^2(t) = 0 \qquad (4)$$

where c_1, c_2, c_3 denote coefficients, ${}_0D^{\alpha_1}$ and ${}_0D^{\alpha_2}$ are fractional derivatives defined by formula (3) and α_1, α_2 denote real orders of fractional derivatives which satisfy the following conditions: $\alpha_1, \alpha_2 \in [0, 1)$ and $\alpha_1 \geq \alpha_2$. In this equation we assumed the same initial conditions as for Eqn. (1). Eqn. (4) may solve numerically due to its nonlinear form.

Fig. 1 shows an example of analytical and numerical solutions of Eqns (1) and (2) respectively.

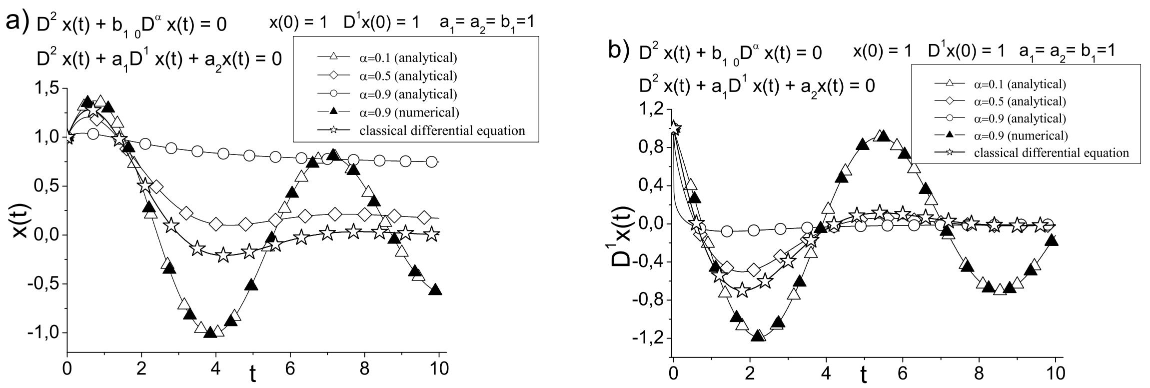

Figure 1: Examples of solutions of Eqns (1) and (2): a) the function $x(t)$ over time; b) the first derivative $D^1x(t)$ over time.

Analyzing this figure we can say that the solution obtained from Eqn. (2), where the parameter α varies between 0 and 1, is general in comparison to the solution obtained by Eqn. (1). It should be noted that both solutions are quite similar for $\alpha = 0.5$. Moreover, using own numerical procedure described in [3] we observe that numerical results quite good reflects the analytical one.

During full presentation we will focus on some details of our numerical treatment and we will explain how to solve numerically Eqn. (4).

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