Pairing and bound states in fermionic systems

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Recent Progress in Many Body Theories 14, Barcelona, July 17

Outline

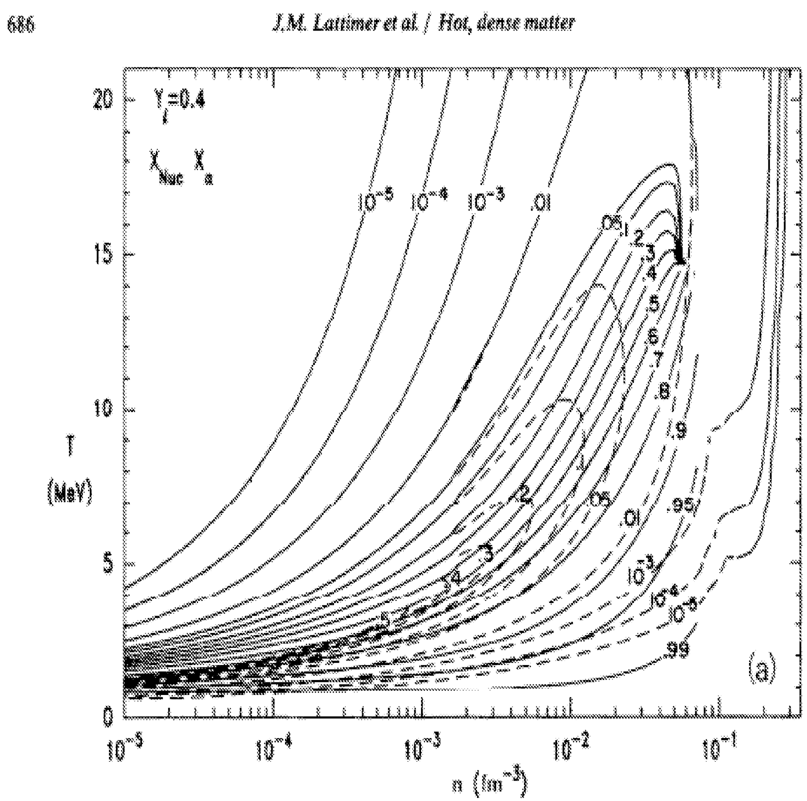
- Alpha particle condensation in dense matter
- Few-body bound states in nuclear medium
- Pairing in nuclear systems
- Weak interactions and neutrino rates

Alpha condensation

- Phenomenology of alpha condensation:
  - The excited states of $4N$ nuclei are well described within the α particle model: elementary degrees of freedom are α's interacting via a α-α potential: ^{8}Be (unstable) ^{12}C (first stable α nucleus), ^{16}O, ^{40}Ca.
  - Recent work suggest that these systems are well described by single wave function (BEC in systems with a few particles ?).
  - This motivates the study of Bose-Einstein condensation in infinite alpha matter - start with $N \to \infty$ system and follow the crossover as N is reduced.

The phase diagram of supernova matter

- Densities $\rho \sim 10^{12}$ g cm^{-3}; Temperatures $T \leq 10$ MeV; content $\rightarrow 15 - 20\%$ α particles



- Quantum degeneracy (BEC)
- Opacities
- Neutrino sphere and signal

From Hamiltonian to effective action

- The non-relativistic theory of bosons can be mapped onto relativistic $O(n)$ model when $T \to T_c^+$

$$
\begin{aligned}
H &= \int d^3x \left[\frac{\hbar^2}{2m} \nabla\psi^\dagger(\mathbf{x})\nabla\psi(\mathbf{x}) - \mu\psi^\dagger(\mathbf{x})\psi(\mathbf{x}) \right. \\
&\qquad \left. + \psi^\dagger(\mathbf{x})\psi(\mathbf{x})U(\mathbf{x}) \right], \\
U(\mathbf{x}) &= \int d\mathbf{x}' V_2(\mathbf{x}', \mathbf{x})\psi^\dagger(\mathbf{x}')\psi(\mathbf{x}') \\
&+ \int d\mathbf{x}' \int d\mathbf{x}'' V_3(\mathbf{x}, \mathbf{x}', \mathbf{x}'')\psi^\dagger(\mathbf{x}'')\psi(\mathbf{x}')\psi^\dagger(\mathbf{x}'')\psi(\mathbf{x}'),
\end{aligned}
$$

- Expand the fields ψ and ψ in Matsubara sums and keep near T_c the zeroth order term (Baym, Blaizot, Zinn-Justin, 1999, Zinn-Justin 1989).
- Introduce new real fields: $\psi_\pm = \eta(\phi_1 \pm i\phi_2)$.
- The resulting continuum action describes a classical $O(2)$ symmetric scalar ϕ^6 field theory in 3d:

$$\mathcal{S}(\phi) = \int d^3x \Bigg\{ \frac{1}{2} \sum_\nu \left[\partial_\nu \phi(\mathbf{x})\right]^2 + \frac{r}{2}\phi(\mathbf{x})^2$$

$$\frac{u}{4!}\left[\phi(\mathbf{x})^2\right]^2 \frac{w}{6!}\left[\phi(\mathbf{x})^2\right]^3 \Bigg\}.$$

- Negative quartic, and positive sextic terms!

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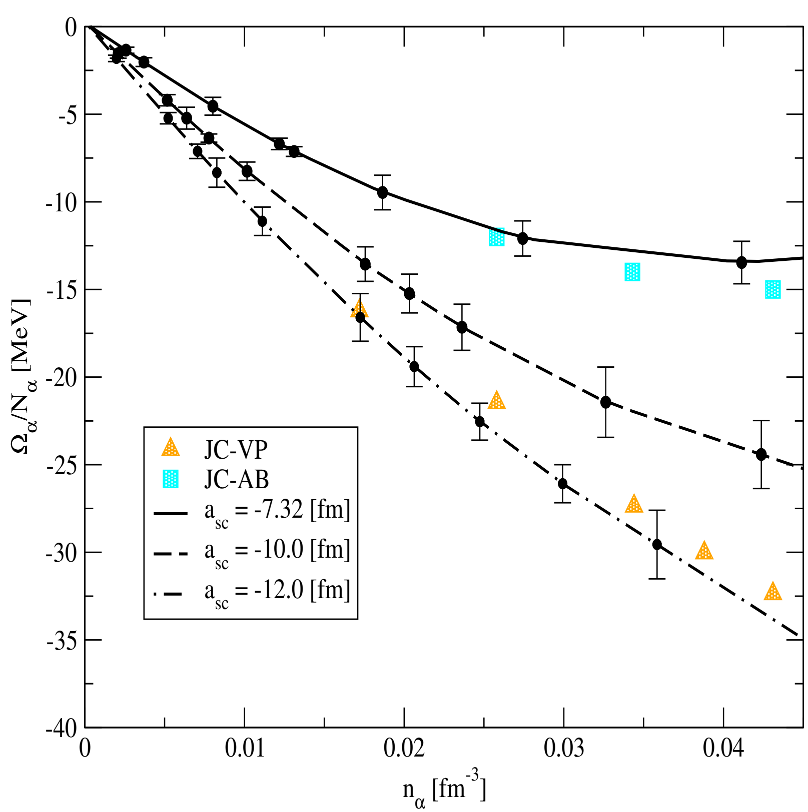
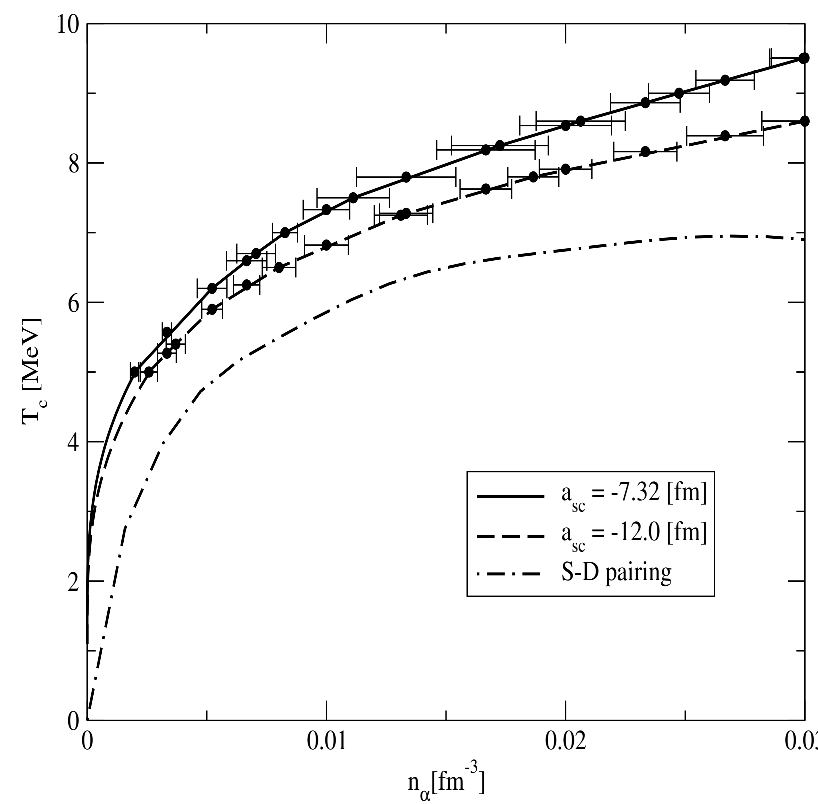
- Negative quartic, and positive sextic terms!

Partition function

- Compute the partition function on 3d spatial lattice

$$\mathcal{Z} = \int [d\phi(\mathbf{x})] \exp\left[-\mathcal{S}(\phi)\right]$$





- Results could be supplemented by an HNC4/HNC5 calculations at $T = 0$ of condensate fraction.
- Potentially rich astrophysical implications of alpha condensations remain be studied
- What about the other light clusters $A = 2, 3$?
  - The deuteron and triton are less bound (-2.2 MeV) and (-8.4 MeV) and thus less stable.
  - Weakly bound states are ideal sources of opacity! example: H^- opacity is dominant in ordinary stars.

The three-body problem in background medium

- Work within the Kadanoff-Baym real-time GF method $\rightarrow$ two and three-point functions are defined on the Schwinger-Keldysh contour
- The three-body equation for the $\mathcal{T}$-matrix

$$\mathcal{T} = \mathcal{V} + \mathcal{V}\,\mathcal{G}\,\mathcal{V} = \mathcal{V} + \mathcal{V}\,\mathcal{G}_0\,\mathcal{T},$$

  where the interaction $\mathcal{V} = V_{12} + V_{23} + V_{13}$

- Reformulate the problem: $\mathcal{T} = \mathcal{T}^{(1)} + \mathcal{T}^{(2)} + \mathcal{T}^{(3)}$

$$\mathcal{T}^{(k)} = \mathcal{V}_{ij} + \mathcal{V}_{ij}\mathcal{G}_0\mathcal{T} \quad ijk = 123,\ 231,\ 312.$$

  Define: $\mathcal{T}_{ij} = \mathcal{V}_{ij} + \mathcal{V}_{ij}\mathcal{G}_0\mathcal{T}_{ij}$ and eliminate the potentials

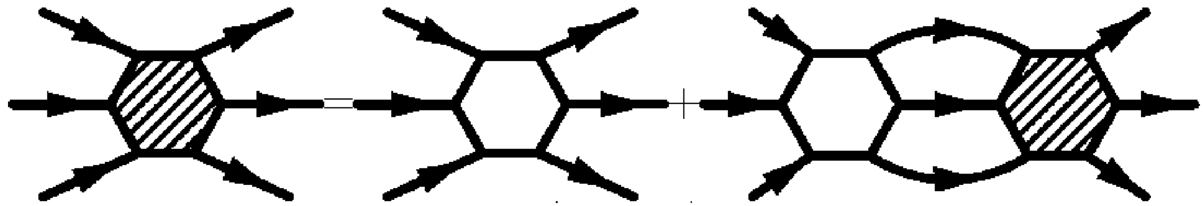
- Non-singular three-body equations (Bethe-Faddeev) with the time structure of the three-body T-matrix

$$\mathcal{T}^{R\,(1)}(t,t') = \mathcal{T}^{R}_{23}(t,t') + \int \left[\, \mathcal{T}^{R\,(2)}(t,\bar{t}) + \mathcal{T}^{R\,(3)}(t,\bar{t})\right] \mathcal{G}^{R}_{0}(\bar{t},t'')\mathcal{T}^{R}_{23}(t'',t')d\bar{t}dt'',$$

- Many particle-hole channels (new aspect of medium physics)

$$\mathcal{G}^{R}_{0}(t_1,t_2) = \theta(t_1-t_2)\begin{cases} G^{>}G^{>}G^{>}(t_1,t_2)-(>\leftrightarrow<) & (3p) \\ G^{>}G^{>}G^{<}(t_1,t_2)-(>\leftrightarrow<) & (2ph) \\ G^{>}G^{<}G^{<}(t_1,,t_2)-(>\leftrightarrow<) & (p2h) \\ G^{<}G^{<}G^{<}(t_1,t_2)-(>\leftrightarrow<) & (3h) \end{cases}$$

Particle-hole content of the T-matrix



- 3-particle – 3-hole scattering T-matrix

$$\mathcal{T}^{R\,(1)} = \mathcal{T}^R_{23} + \int \left[\mathcal{T}^{R\,(2)} + \mathcal{T}^{R\,(3)} \right] \frac{Q_3(\Omega')}{\Omega - \Omega' + i\eta} \mathcal{T}^R_{23}(\Omega') d\Omega',$$

- 3-body Pauli-blocking: $\bar{f}_F = 1 - f_F$

$$Q_3(p_\alpha, p_\beta, p_\gamma) = \bar{f}_F(p_\alpha)\bar{f}_F(p_\beta)\bar{f}_F(p_\gamma) - f_F(p_\alpha)f_F(p_\beta)f_F(p_\gamma).$$

p_α are spanned in terms of Jacobi coordinates.

Bound states in background medium

- Bound state wave-function

$$\Psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)}; \quad \psi^{(k)} = \mathcal{G}_0 T_{ij}(\psi^{(i)} + \psi^{(j)}). \quad \text{(-8)}$$

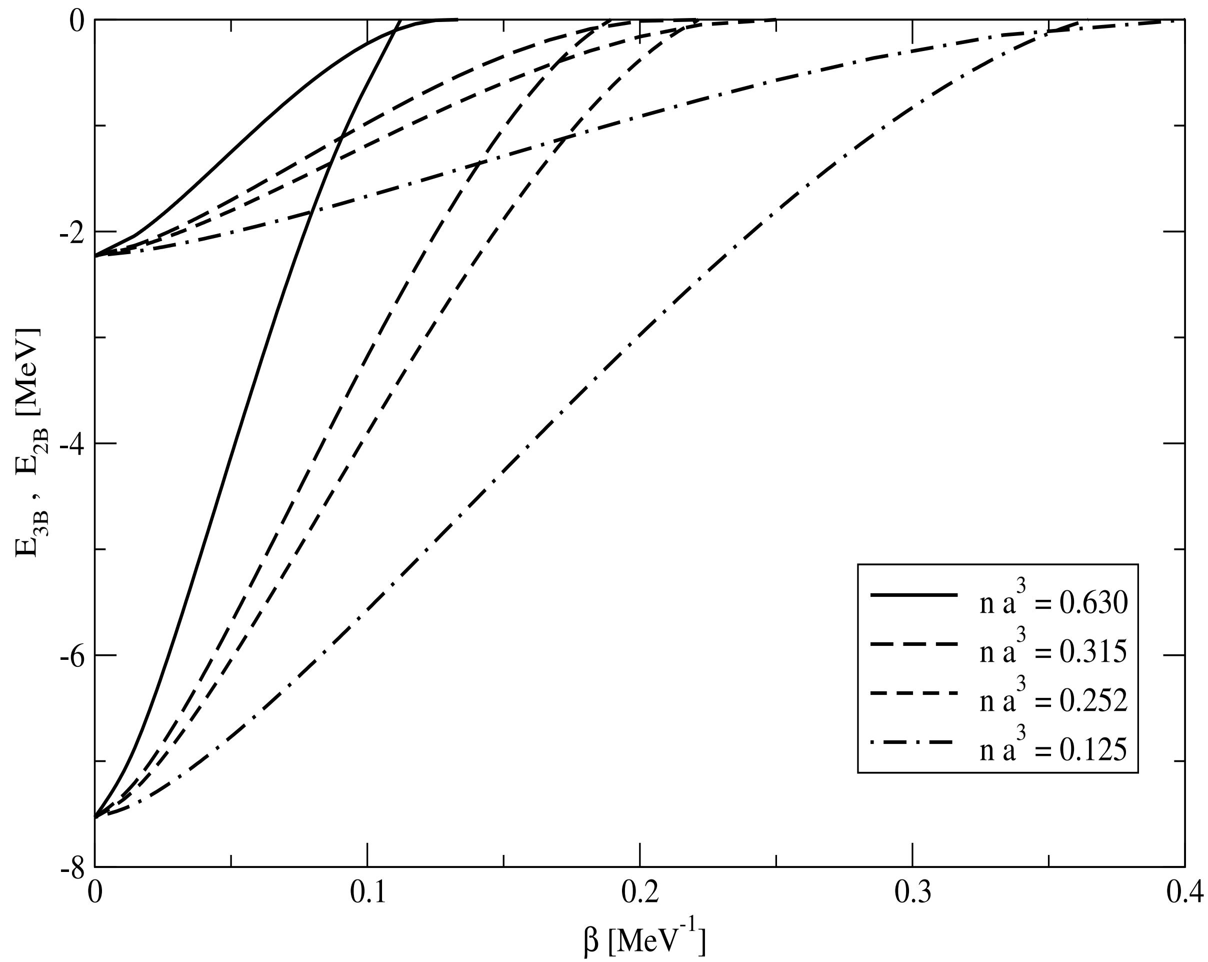
- Need the channel T-matrix

$$T^R(\vec{p}, \vec{p}'; \vec{P}, E)$$
$$= V(\vec{p}, \vec{p}') + \int \frac{d\vec{p}''}{(2\pi)^3} V(\vec{p}, \vec{p}'') G_0^R(\vec{p}'', \vec{P}, E) T^R(\vec{p}'', \vec{p}'; \vec{P}, E)$$

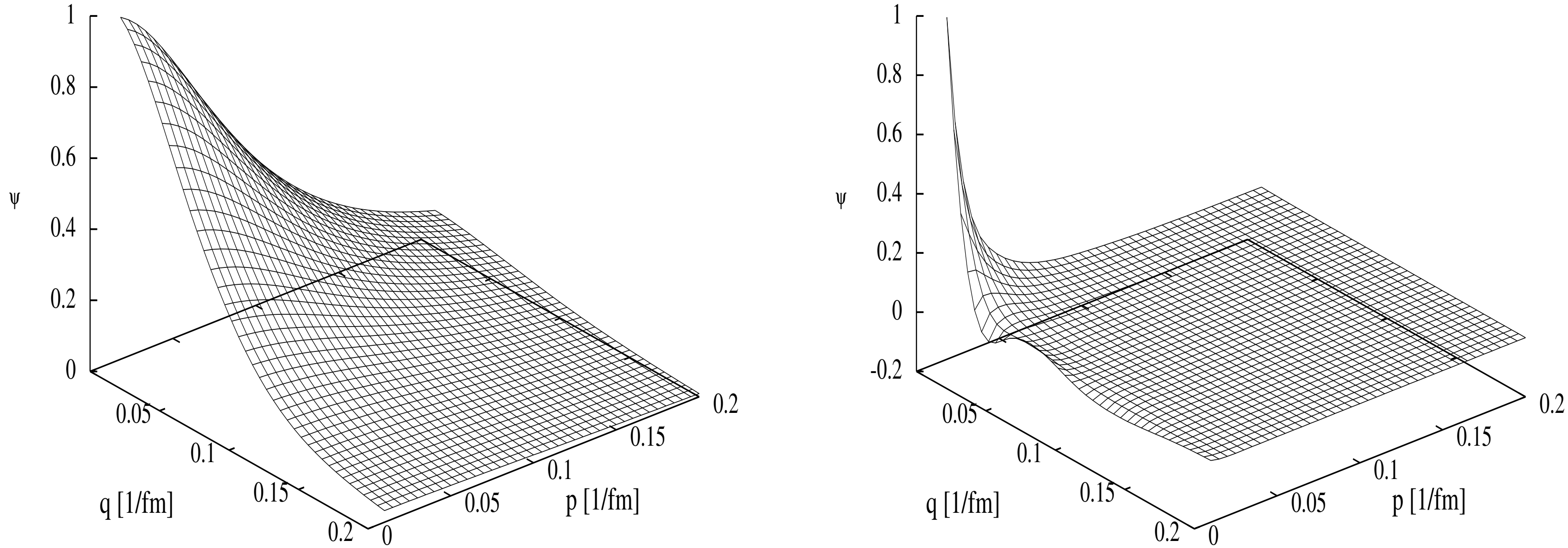
$$G_0^R(\vec{k}_1, \vec{k}_2, E) = \frac{Q_2(\vec{k}_1, \vec{k}_2)}{E - \epsilon(\vec{k}_1) - \epsilon(\vec{k}_2) + i\eta}, \quad \text{(-10)}$$

Poles $\rightarrow A = 2$ binding energies $E_B(T, \rho)$.

Background dependent binding energies of triton in nuclear matter



The ratio $\eta = E_{3B}(T)/E_{2B}(T)$ is independent of temperature.



Squeezing the wave function in the momentum space: signal of a quantum phase transition to the unbound state.

What happens beyond the extinction point?

- The deuteron crosses over to become a Cooper pair
- Precritical three-body correlations for $T > T_c$?

Nozieres-Schmitt-Rink solutions

The evolution deuteron $\rightarrow$ Cooper pair is well described by the mean-field BCS theory (coupled $^3S_1 -^3 D_1$ channels)

$$\Delta^{(i)}(p,\Lambda) = \int^{\Lambda} \frac{dp'p'^2}{(2\pi)^3} V^{3SD1}(p,p') \times \frac{\Delta^{(i-1)}(p',\Lambda)}{\sqrt{E_p^2 + D^{(i-1)}(p',\Lambda)^2}} [f(\omega_+) - f(\omega_-)]. \quad \text{(-11)}$$

$$n = 4 \int \frac{d^3p}{(2\pi)^3} \left[u_p^2 f(\omega_+) + v_p^2 f(\omega_-) \right]. \quad \text{(-11)}$$

Solved iteratively with a running cut-off Λ; the iteration is stopped when $d\Delta/d\Lambda = 0$.

BCS-BEC crossover

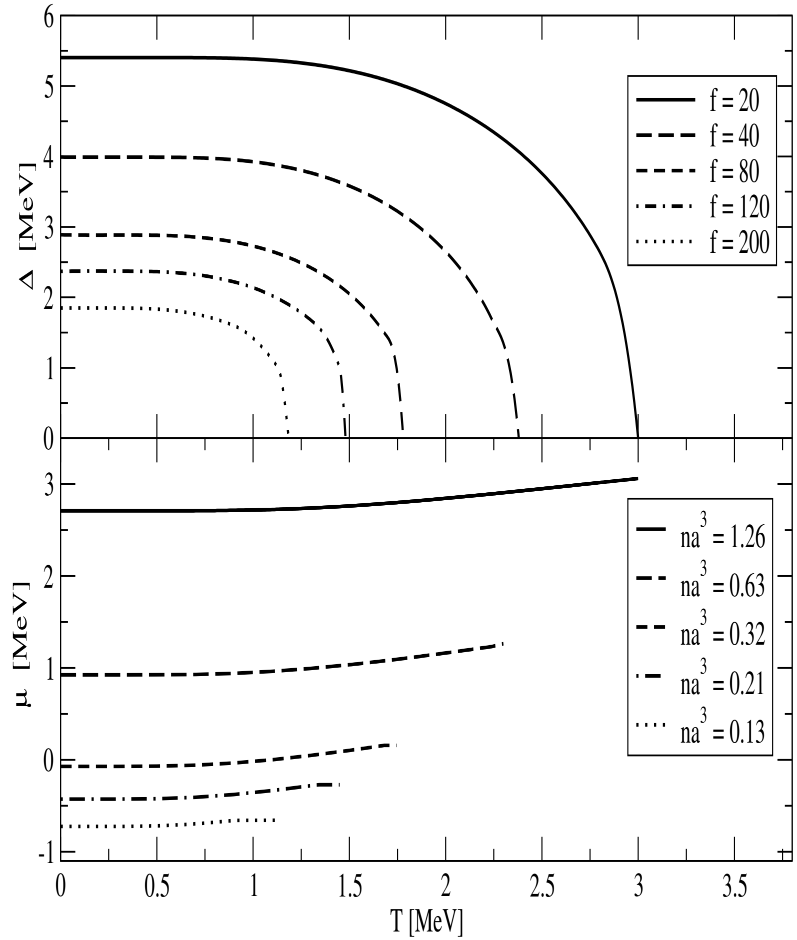
- Nozières-Schmitt-Rink conjecture: the BCS theory smoothly interpolates between the weak and strong couplings
- In the BEC limit the pair-wave function goes over to the Schrödinger equation

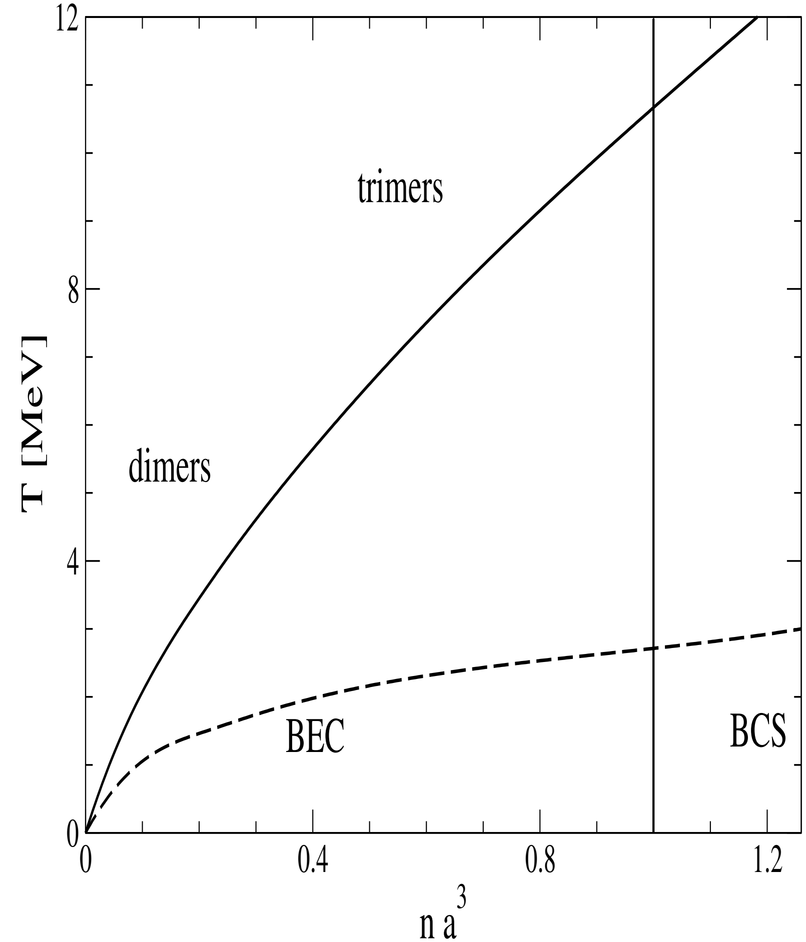
$$\psi(k) = \langle a^{\dagger}_{n,\vec{k}} a^{\dagger}_{p,-\vec{k}} \rangle = \frac{\Delta(k)}{2E_k} \left[1 - f(E_k^+) - f(E_k^-)\right],$$

$$\frac{k^2}{m}\psi(k) + \left[1 - f(E_k^+) - f(E_k^-)\right] \sum_{k'} V(k,k')\psi_{l'}(k') = 2\mu\,\psi(k)$$

- In unbalanced systems phases with broken space symmetries intervene.

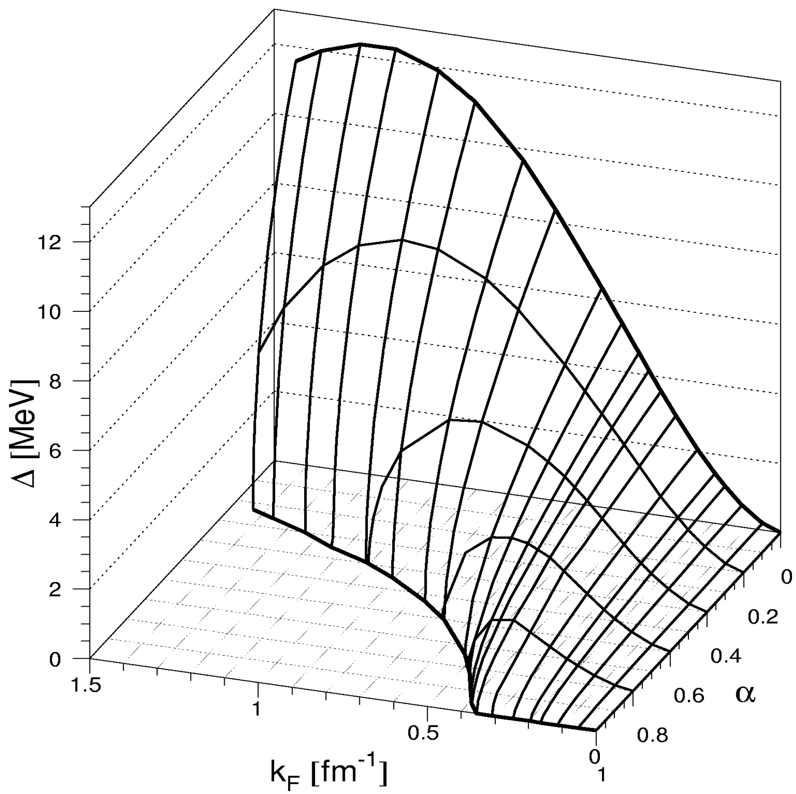
BCS-BEC crossover

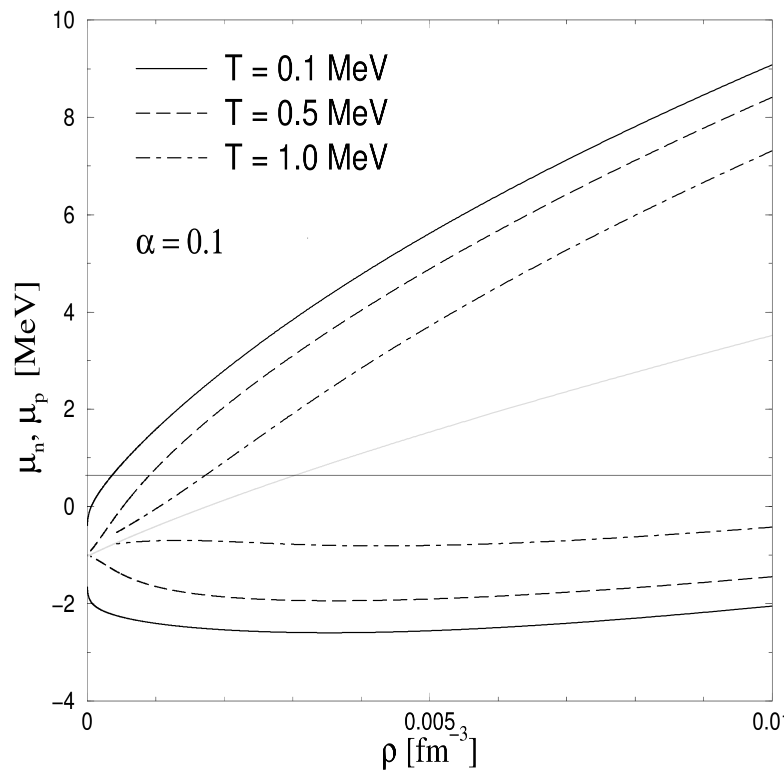




At the cross-over μ sign changes, coherence length $\xi \sim n^{1/3}$.

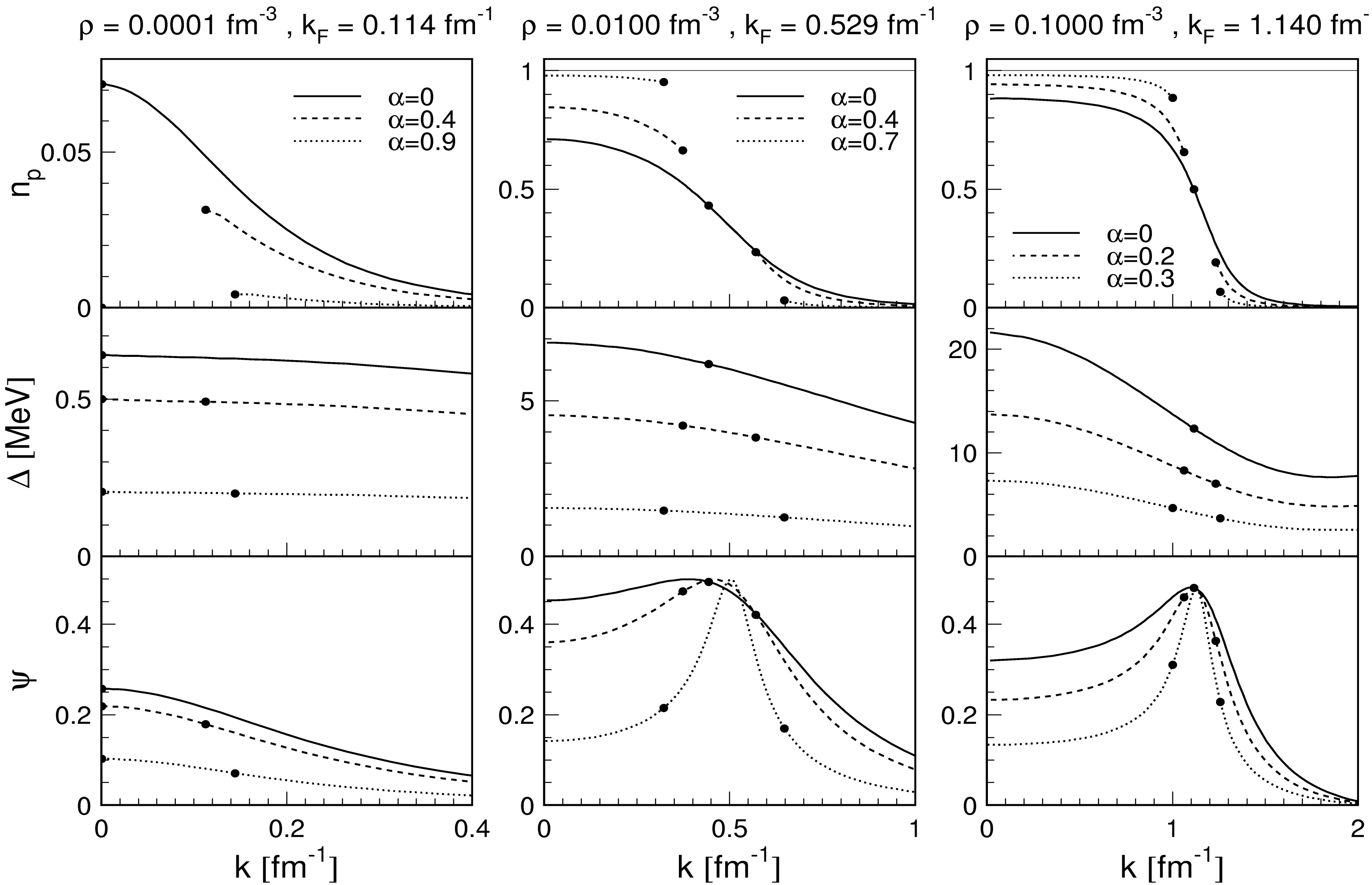
BCS-BEC crossover in asymmetric systems



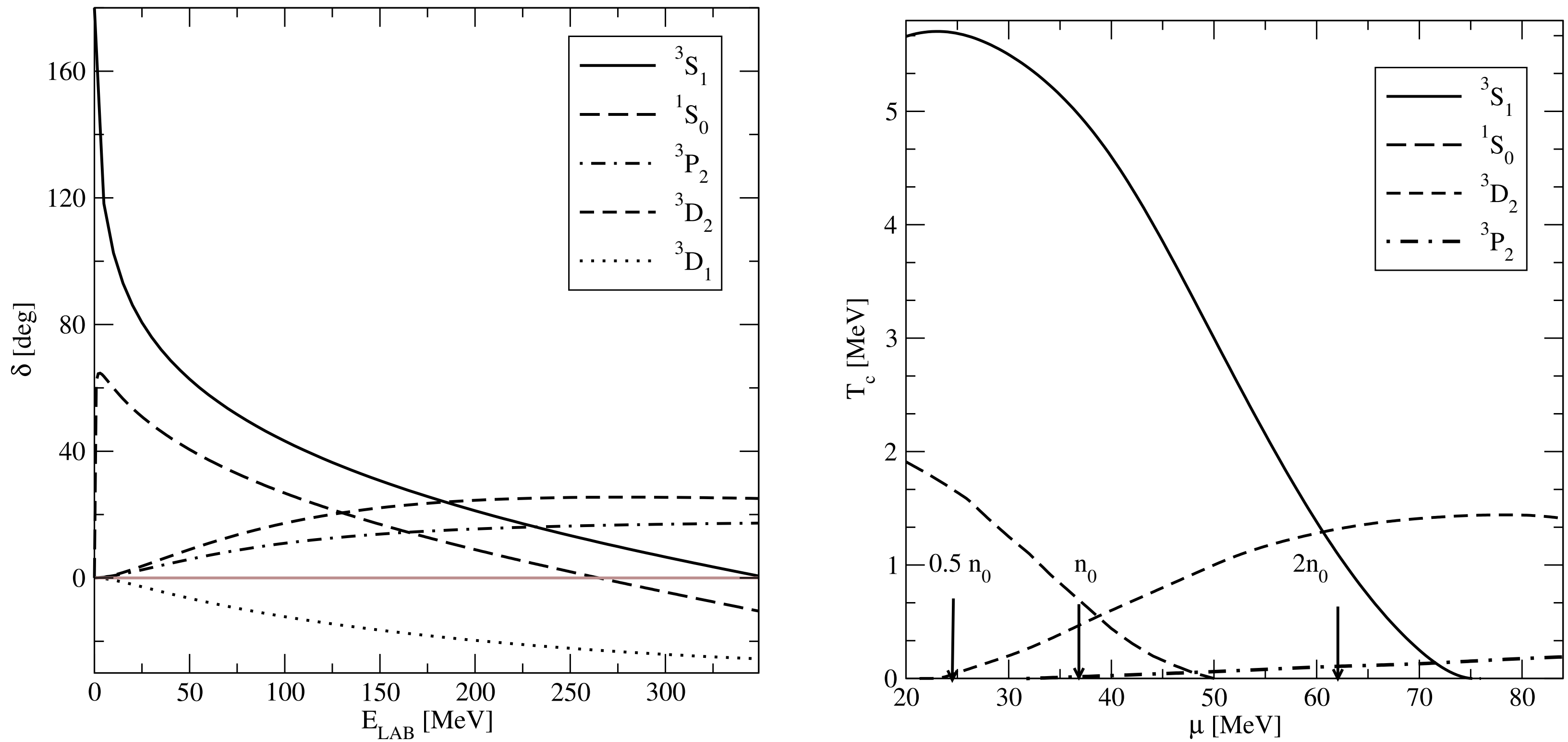


U. Lombardo, P. Nozieres, P. Schuck, H.-J. Schulze, A. Sedrakian, Phys.Rev. C64 (2001) 064314

Crossover continued



Nuclear systems



Left panel. Dependence of the experimental scattering phase shifts in the $^3S_1$, $^3P_2$, $^3D_2$, and $^3D_1$ partial waves on the laboratory energy. *Right panel.* The dependence of the critical temperatures of superfluid phase transitions in the attractive channels on the chemical potential.

- Pairing in neutron matter with retardation - strong coupling Eliashberg theory
  - time-non-local interactions - gaps are then energy dependent
  - keep same approximations for normal and anomalous self-energies (Ward identities)
  - possible in medium modifications of the meson dynamics (e.g. precursor of pion condensation).
- Long + short range components:

$$
\begin{aligned}
H_{\pi NN} &= -\frac{f_\pi}{m_\pi}(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})(\boldsymbol{\tau}\cdot\phi) \\
&+ \Big[g'\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + h'\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\, S_{12}(\boldsymbol{n})\Big]\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2,
\end{aligned}
$$

DSE in nucleonic sector

- Normal propagators:

$$G^R(p) = \frac{\omega Z(p) + \xi_p^*}{(\omega + i\eta)^2 Z(p)^2 - \xi_p^{*2} - \Delta^R(p)^2},$$

- Anomalous propagators:

$$F^R(p) = -\frac{\Delta^R(p)}{(\omega + i\eta)^2 Z(p)^2 - \xi_p^{*2} - \Delta^R(p)^2},$$

- Normal self-energies: $\Sigma^R = \Sigma_S^R + \Sigma_A^R$

$$\xi^*(p) = \xi(p) + \Sigma_S^R(p)$$

- Wave-function renormalization: $Z(p) = 1 - \omega^{-1}\Sigma_A^R(\omega)$.

DSE in bosonic sector

- DSE for mesons: $\hat{D}(p) = \hat{D}_0(p) + \hat{D}_0(p)\hat{\Pi}(p)\hat{D}(p)$.
- Polarization tensor and the vertex:

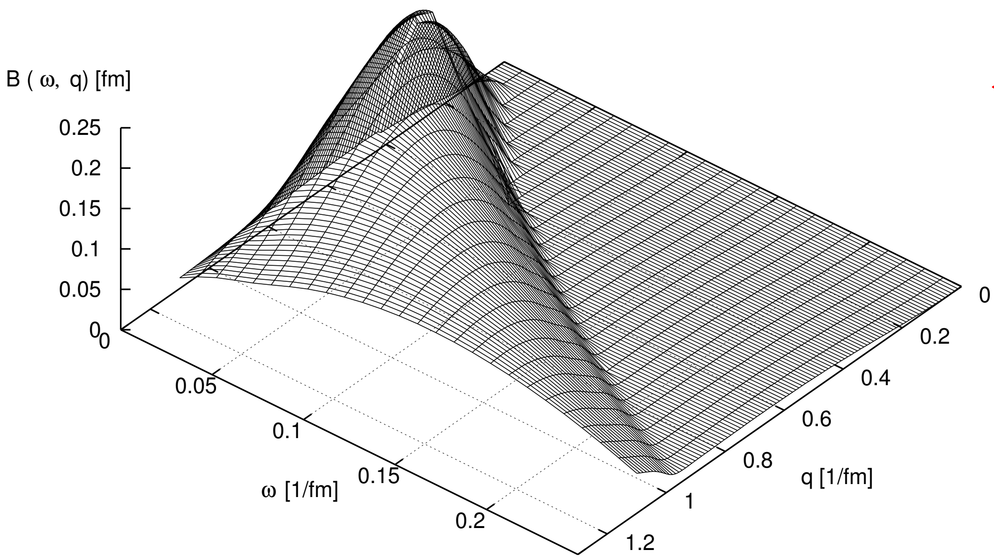
$$\hat{\Pi}(q) = -\mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\, i\,\hat{\Gamma}_0(q)\hat{G}(p+q)\hat{G}(-p)\hat{\Gamma}(q),$$

$$\hat{\Gamma}(q) = \hat{\Gamma}_0(q) + \mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\, i\,\hat{\Gamma}_1(q)\hat{G}(p+q)\hat{G}(-p)\hat{\Gamma}(q).$$

- Pion spectral function:

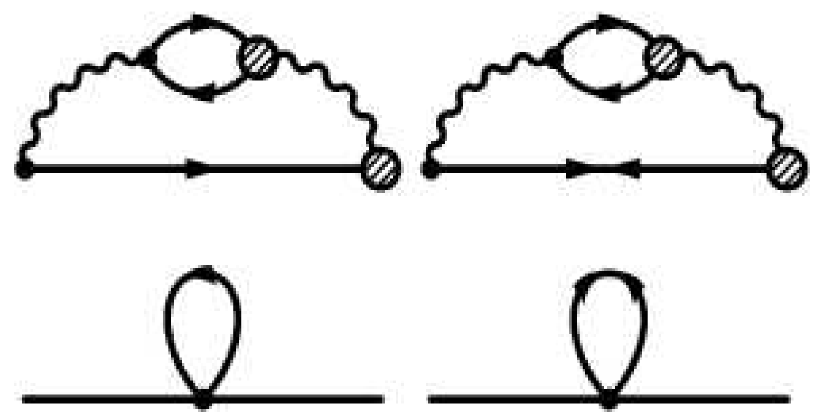
$$B(q) = \frac{-2\mathrm{Im}\Pi^R(q)}{[\omega^2 - \vec{q}^2 - m_\pi^2 - \mathrm{Re}\Pi^R(q)]^2 + [\mathrm{Im}\Pi^R(q)]^2}.$$

Pion spectral function



← Pion spectral function in cold neutron matter

Hartree and Fock contributions to the self-energies of neutrons→



Retarded (Fock) part of the self-energies

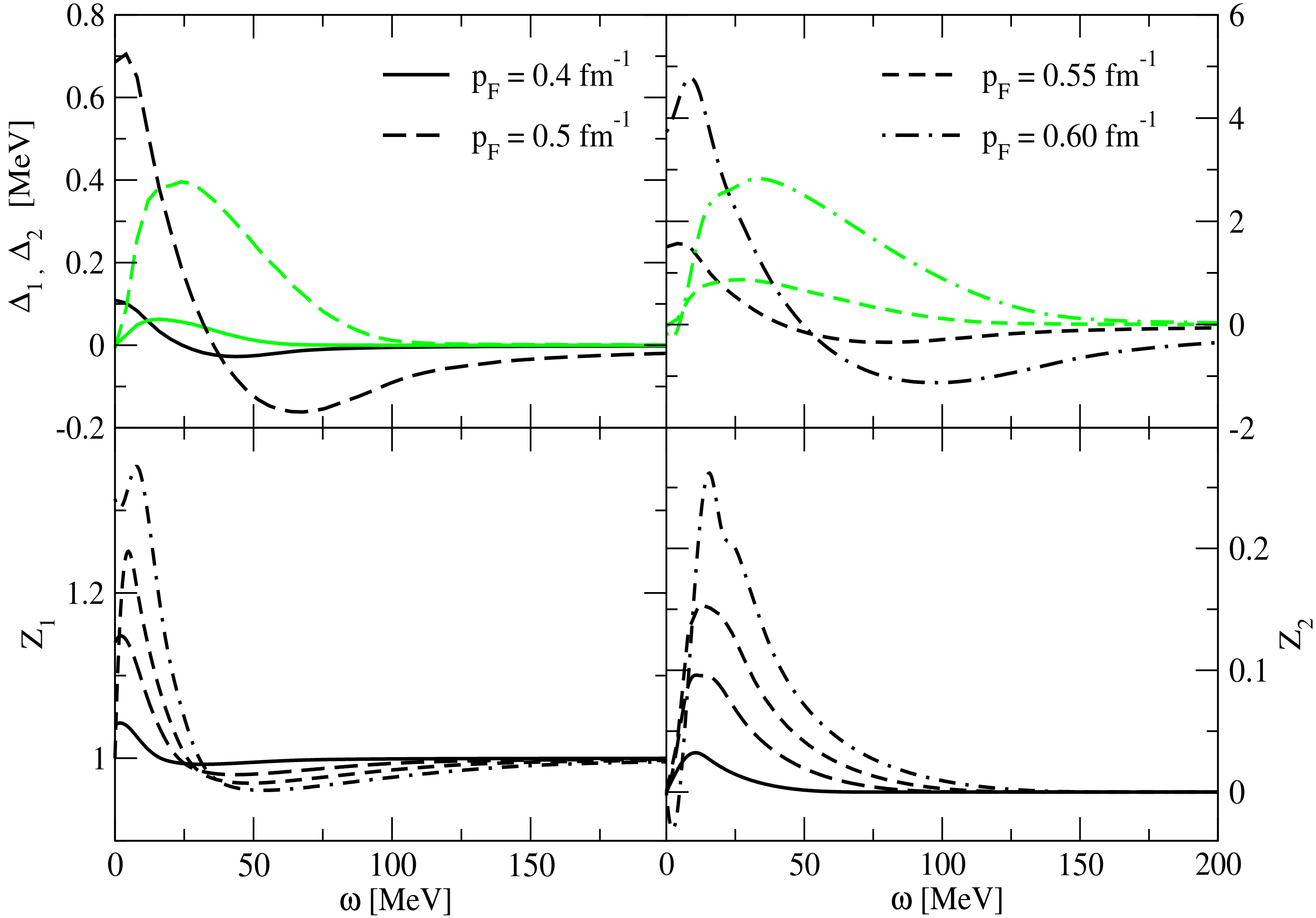
$$\Sigma^R(\omega,\vec{p}) = \mathrm{Tr}\int\frac{d^3q d\varepsilon}{(2\pi)^4}\Gamma_0(\vec{q})A_G(\varepsilon,\vec{p}-\vec{q})C(\omega,\varepsilon,\vec{q})\Gamma(\vec{q}),$$

$$\Delta^R(\omega,\vec{p}) = \mathrm{Tr}\int\frac{d^3q d\varepsilon}{(2\pi)^4}\Gamma_0(\vec{q})A_F(\varepsilon,\vec{p}-\vec{q})C(\omega,\varepsilon,\vec{q})\Gamma(\vec{q}),$$

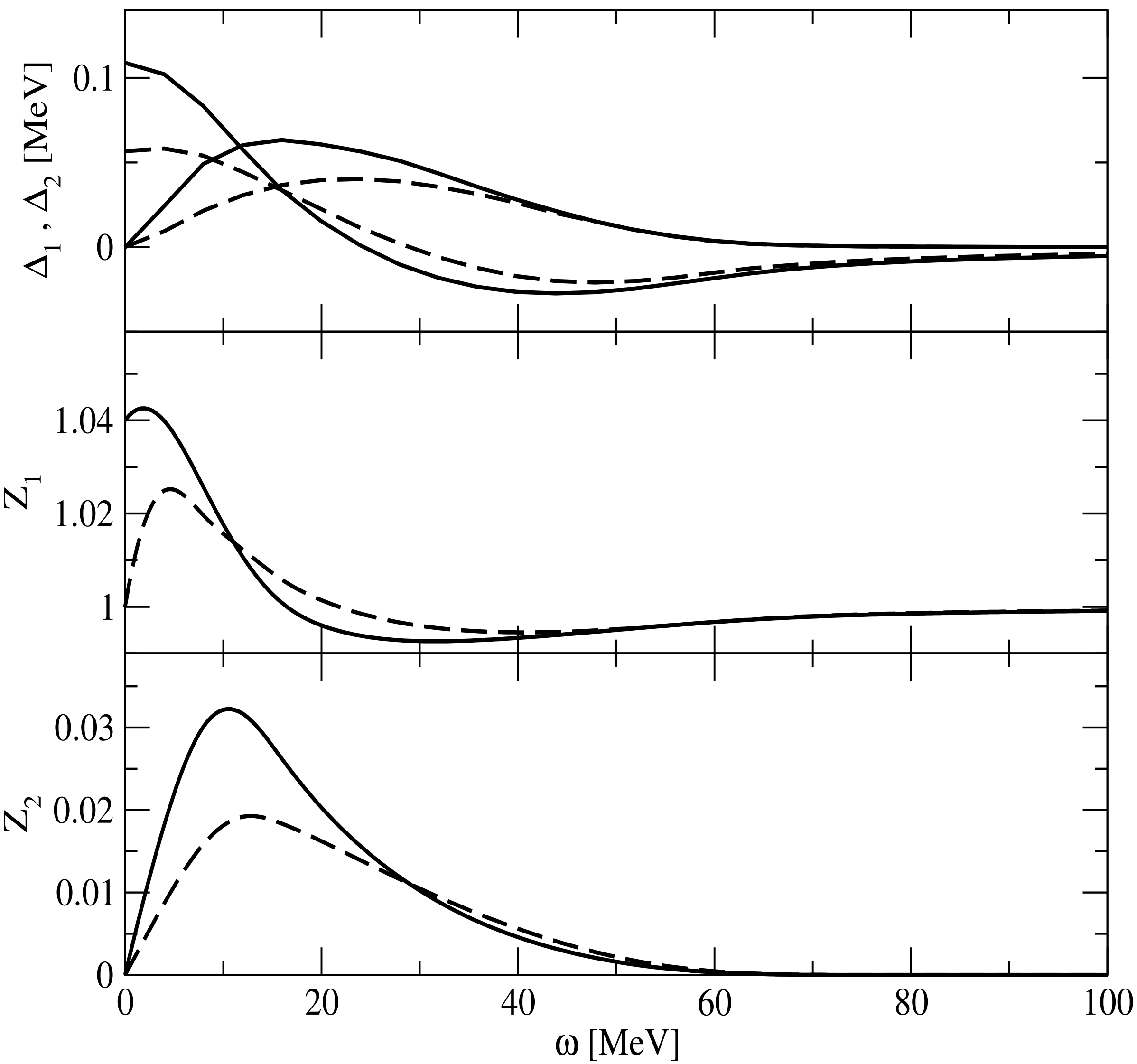
$$C = \int_0^\infty\frac{d\omega'}{2\pi}B(\omega',\vec{q})\left[\frac{f(\varepsilon)+g(\omega')}{\varepsilon-\omega'-\omega-i\eta}+\frac{1-f(\varepsilon)+g(\omega')}{\varepsilon+\omega'-\omega-i\eta}\right]$$

- Need to solve self-consistently 4 integral equations (= 2 complex equation for the normal and anomalous self-energies).
- The kernels of these integral equations are singular either at the boundary or within the integration range.
- Iterative procedure; good convergence after 15-20 iterations.

Off-shell characteristics: gap and wave-function renormalization



Different approximations to the spectral function



Effects of pairing on neutrino radiation from NS

Processes on fermions

- Neutral current processes (Z_0 exchange)

$$\begin{cases} f_1 \longrightarrow f_2 + \nu_f + \bar{\nu}_f & \text{(brems)} \\ f_1 + f_1' \longrightarrow f_2 + f_2' + \nu_f + \bar{\nu}_f \end{cases} \qquad (-21)$$

- Charged current processes ($W^{\pm}$ exchange)

$$\begin{cases} f_1 \longrightarrow f_2 + e + \bar{\nu}_e & \text{(Urca)} \\ f_1 + f_1' \longrightarrow f_2 + f_2' + e + \bar{\nu}_e \end{cases} \qquad (-22)$$

Transport equations

- ν and $\bar{\nu}$ - Boltzmann equations with KB collision integrals

$$\left[\partial_t + \vec{\partial}_q\, \omega_\nu(\vec{q})\vec{\partial}_x\right] f_\nu(\vec{q},x)$$
$$= \int_0^\infty \frac{dq_0}{2\pi} \mathrm{Tr}\left[\Omega^<(q,x)S_0^>(q,x) - \Omega^>(q,x)S_0^<(q,x)\right],$$

- ν-quasiparticle propagators:

$$S_0^<(q,x) = \frac{i\pi\, \not{q}}{\omega_\nu(\vec{q})}\Big[\delta\left(q_0 - \omega_\nu(\vec{q})\right) f_\nu(q,x)$$
$$-\delta\left(q_0 + \omega_\nu(\vec{q})\right)\left(1 - f_{\bar{\nu}}(-q,x)\right)\Big].$$

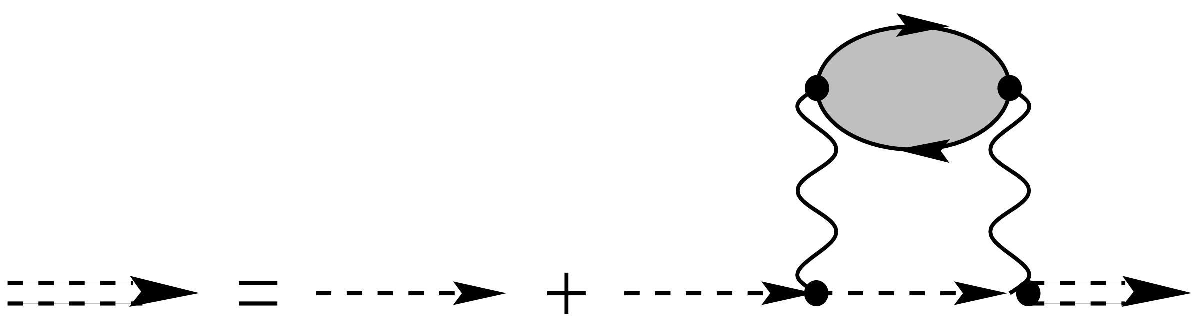
- definition of the Poisson bracket

$$\{f,g\}_{P.B.} = \partial_\omega f\ \partial_t g - \partial_t f\ \partial_\omega g - \partial_{\vec{p}} f\ \partial_{\vec{r}} g + \partial_{\vec{r}} f\ \partial_{\vec{p}} g.$$

Self-energies

- ν and $\bar{\nu}$-self-energies (second order in weak force)

$$-i\Omega^{>,<}(q_1,x) = \int \frac{d^4q}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}(2\pi)^4\delta^4(q_1-q_2-q)$$
$$i\Gamma^{\mu}_{L\,q}\, iS_0^{<}(q_2,x) i\Gamma^{\dagger\,\lambda}_{L\,q} i\Pi^{>,<}_{\mu\lambda}(q,x), \quad \text{(-26)}$$

- 

- the central problem is to compute the polarization tensor!

Bremsstrahlung emissivity

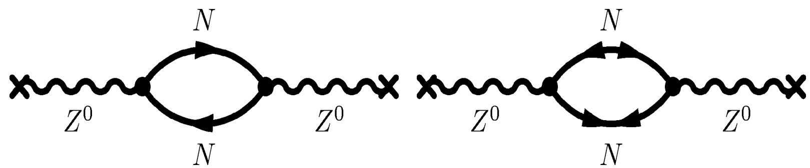
- energy loss per unit time and volume

$$\epsilon_{\nu\bar{\nu}} = \frac{d}{dt}\int \frac{d^3q}{(2\pi)^3}\left[f_\nu(\vec{q}) + f_{\bar{\nu}}(\vec{q})\right]\omega_\nu(\vec{q}) \tag{-27}$$

- expressed through the collision integrals

$$\epsilon_{\nu\bar{\nu}} = -2\left(\frac{G}{2\sqrt{2}}\right)^2 \sum_f \int \frac{d^3q_2}{(2\pi)^3 2\omega_\nu(\vec{q}_2)} \int \frac{d^3q_1}{(2\pi)^3 2\omega_\nu(\vec{q}_1)} \int \frac{d^4q}{(2\pi)^4}$$
$$(2\pi)^4\delta^3(\vec{q}_1 + \vec{q}_2 - \vec{q})\delta(\omega_\nu(\vec{q}_1) + \omega_\nu(\vec{q}_2) - q_0)\ \left[\omega_\nu(\vec{q}_1) + \omega_\nu(\vec{q}_2)\right]$$
$$g_B(q_0)\left[1 - f_\nu(\omega_\nu(\vec{q}_1))\right]\left[1 - f_{\bar{\nu}}(\omega_\nu(\vec{q}_2))\right]\Lambda^{\mu\lambda}(q_1, q_2)\Im\mathrm{m}\,\Pi^R_{\mu\lambda}(q).$$

Neutral current pair-breaking processes



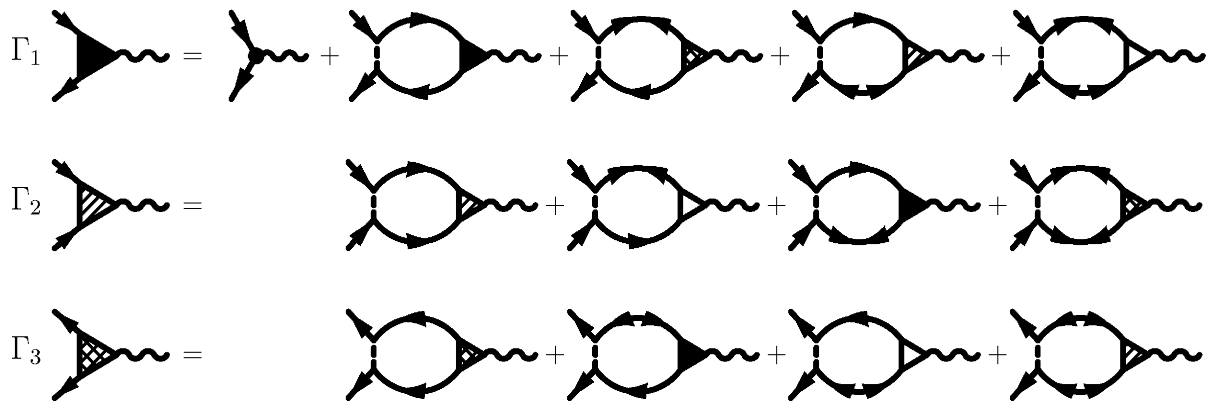
- The one-loop contribution to the polarization tensor in the superfluid matter

$$\epsilon_{\nu\bar{\nu}} = \frac{G^2\, c_V^2}{240\pi^3}\;\nu(p_F)\, T^7\, I(\zeta) \equiv \epsilon_0\, I(\zeta),$$

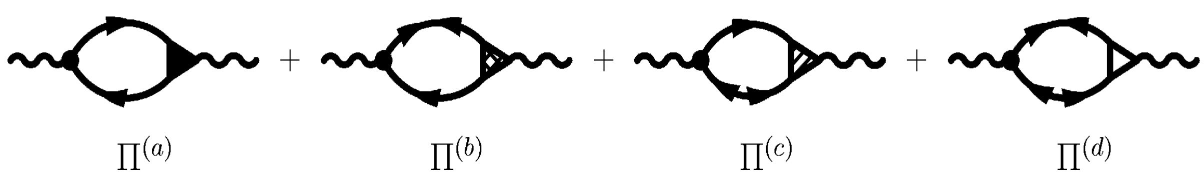
$$I(\zeta) = \zeta^7 \int_0^\infty d\phi\, (\cosh\phi)^5\, f(\zeta\cosh\phi)^2, \quad \zeta = 2\Delta(T)/T$$

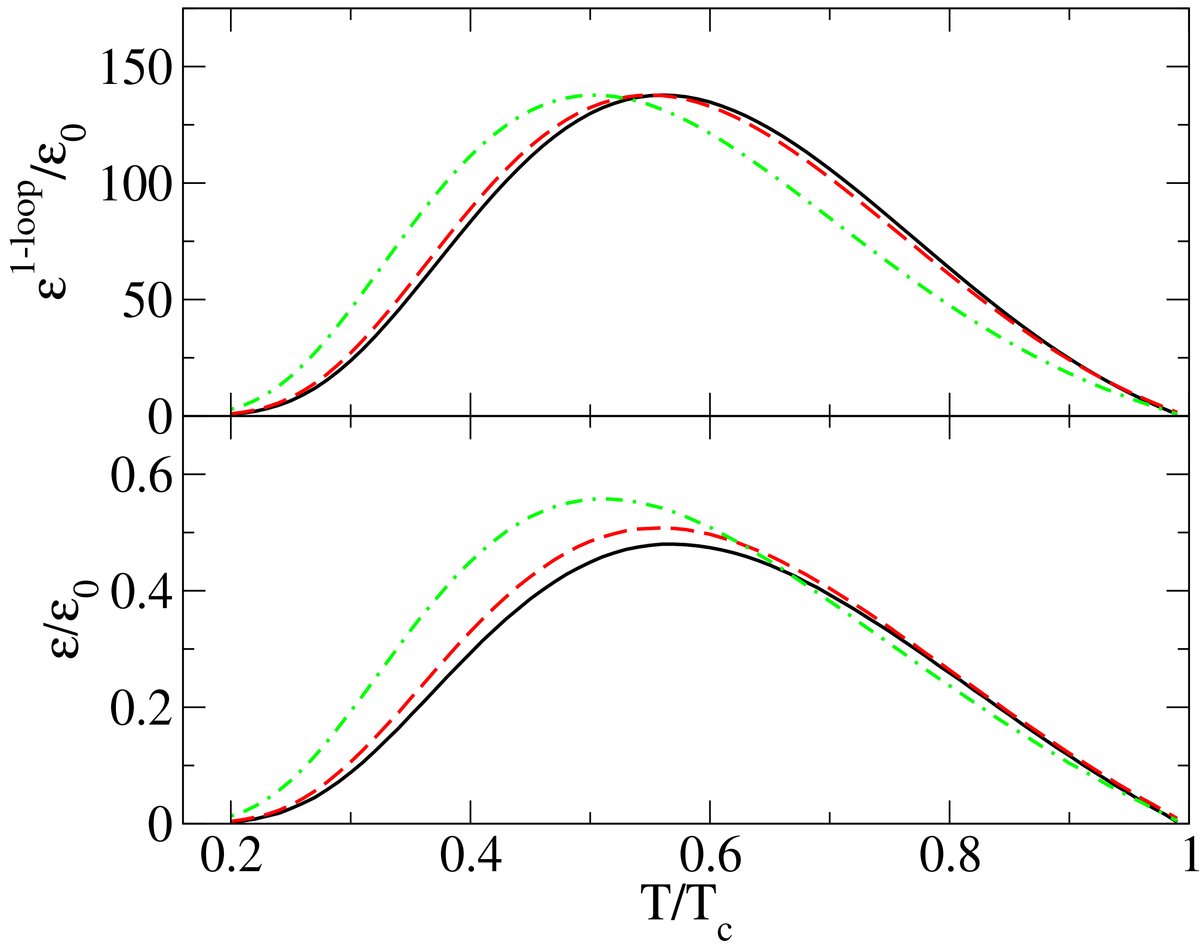
Need to go beyond the one loop - RPA

- The vertex functions in the superfluid phase



- Contributions to the polarization tensor





f-sum rule and gauge invariance

$$\int_{-\infty}^{\infty} d\omega \, \mathrm{Im}\, \Pi^R(\boldsymbol{q}, \omega) = \frac{q^2}{2m}; \quad q^\mu \Pi^R_{\mu\nu}(\boldsymbol{q}, \omega) = 0$$

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