RPMBT14
International Conference on
Recent Progress in Many Body Theories
July 16-20, 2007. Barcelona, Spain

Generalized Entanglement in Static and Dynamic Quantum Phase Transitions

Lorenza Viola
Lorenza.Viola@Dartmouth.edu

People:

Shusa Deng (Dartmouth)
Gerardo Ortiz (Indiana)

Howard Barnum (Los Alamos)
Manny Knill (NIST, Boulder)
Rolando Somma (Los Alamos)

Department of Physics and Astronomy, Dartmouth College

Taming "complexity" in many-body systems...

Goal: To probe, understand, and control quantum phases of matter
– both under equilibrium and nonequilibrium – conditions.

Prerequisite: To obtain qualitative and quantitative understanding of zero-temperature QPTs.

- Conceptual significance:
  - → Central challenge of condensed matter theory, atomic physics, quantum statistical mechanics (coexistence/competition between multiple interactions and quantum orders...)

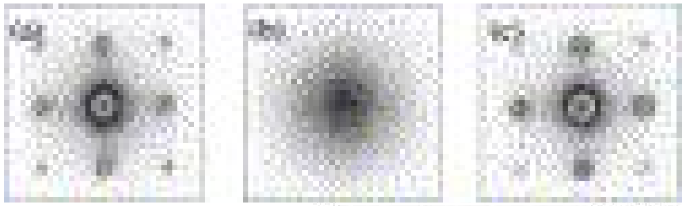
- Practical significance:
  - → Material science and device technology;
  - → Experimental quantum computation and simulation (ultracold atoms in optical lattices...)

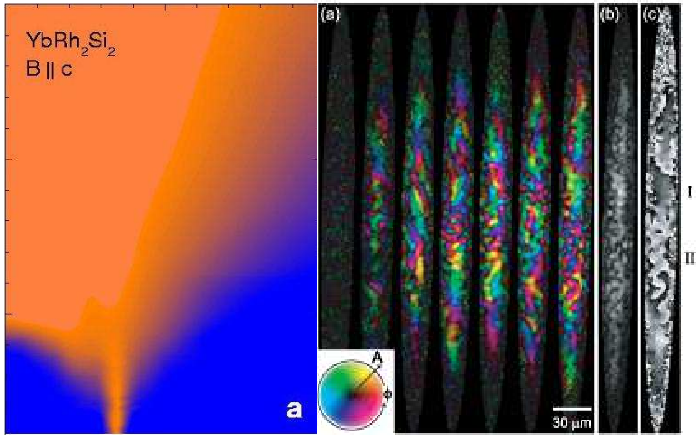
Growing body of experimental work yet theoretical understanding remains poor...

Chief difficulty: complexity of quantum correlations in many-body states and dynamical evolutions...

Can ideas and tools from QIS help?

[Greiner *et al*, Nature 2002]





[Gegenwart *et al*, PRL 2002][Sadler *et al*, Nature 2006]

A natural QIS tool: Entanglement theory

Entanglement is intimately tied to inherent "complexity" of QI processing:

- Can lead to quantum correlations between subsystems that admit no local classical interpretation
- Provides the defining resource for quantum communication (quantum teleportation, superdense coding, communication complexity...)
- Provides a necessary (not sufficient!) resource for pure-state quantum computational speed-up...

Amount of entanglement upper bounded by poly(n) ⇒ *Efficient (poly(n) resources) classical simulatability*

[Josza & Linden, JPA 2002; Vidal, PRL 2003; Datta & Vidal, PRA 2007]

Pay-off for proper accounting of entanglement in many-body systems already impressive:

- Conceptual: Efficient representations of quantum states (MPS, PEPS); Area laws...

[Verstraete & Cirac, cmat/0407066; Eisert & Osborne, PRL 2006]

- Computational: Improved renormalization-group methods for
  - ➞ 1D lattice systems: time-evolving block decimation algorithms;
  - ➞ Higher-dimensional lattice systems: PEPS, entanglement renormalization...

[Verstraete, Porras, Cirac, PRL 2004; Vidal, PRL 2004...]

- Information-theoretic: Computational complexity of variational/DMRG approaches, and PEPS; Efficient solvability of generalized mean-field Hamiltonians...

[Eisert, PRL 2006; Schuch *et al*, PRL 2006; Somma *et al*, PRL 2006]

Entanglement and quantum critical phenomena

Can entanglement theory provide a better understanding of QPTs?

- What is the nature and role of entanglement in a QPT?
- Can entanglement measures detect and classify QCPs?...

[Amico, Fazio, Osterloh, Vedral, RMP, qph/0703044]

Some of the results emerged from extensive analysis of ground-state entanglement:

→ Pairwise entanglement (concurrence) detects QCPs and obeys universal scaling laws in 1D and 2D models...

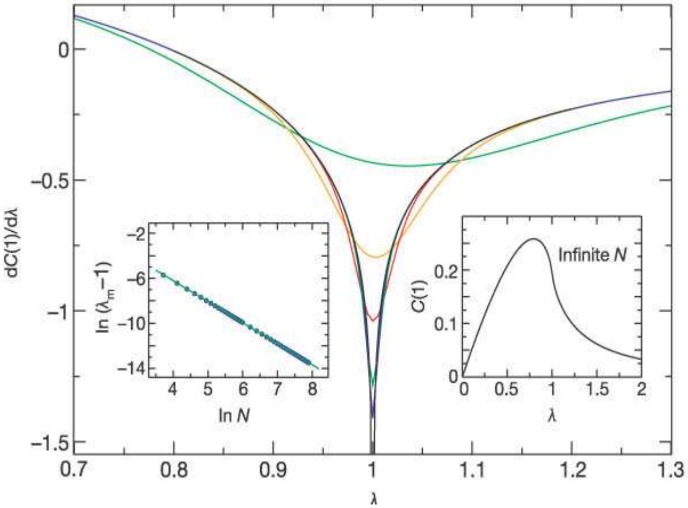
[Osborne & Nielsen, PRA 2002, J Vidal *et al*, PRA 2004; Roscilde *et al*, PRL 2005...]

→ Critical scaling of block entropy agrees with conformal field theory...

[Vidal *et al*, PRL 2003; Latorre *et a*l, QIC 2004...]

→ Localizable entanglement can be long-ranged despite finite correlation length ...

[Verstraete *et al*, PRL 2004...]



[Osterloh *et al*, Nature 2002]

Still, with a few exceptions...

(1) Mostly bipartite entanglement...

(2) Mostly static/equilibrium scenarios...

(3) Mostly distinguishable degrees of freedom...

Is the standard notion of entanglement sufficiently general?...

(Some) limitations of subsystem-based entanglement

A basic fact: **Entanglement is relative...**

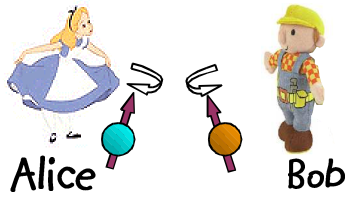
$$\mathcal{H} \simeq \mathcal{H}_A \otimes \mathcal{H}_B$$

$$|Bell\rangle = \frac{|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}}$$

- (Standard) entanglement is un-ambiguously defined only relative to a preferred decomposition of $\mathcal{H}$ into subsystems:

  A pure state in $\mathcal{H}$ is entangled iff it induces **mixed** subsystem states.

- The choice of preferred subsystems is unproblematic in most QIS settings.



What about other physical settings?...

A compelling case: **Quantum many-body systems**

→ How should entanglement be defined for states of indistinguishable particles?

[Eckert *et al*, Ann. Phys. 2002; Zanardi, PRA 2002; Kindermann, PRL 2006; Wolf, PRL 2006; Banuls, Cirac, Wolf, qph/0705.1103...]

$$\langle \vec{r}_1, \vec{r}_2, \ldots | \Psi \rangle \sim \begin{vmatrix} e^{i\vec{k}_1\cdot\vec{r}_1} & e^{i\vec{k}_1\cdot\vec{r}_2} & \ldots \\ e^{i\vec{k}_2\cdot\vec{r}_1} & e^{i\vec{k}_2\cdot\vec{r}_2} & \ldots \\ \vdots & \vdots & \end{vmatrix}$$

→ Particle or mode entanglement? Which set of modes (if any)?

→ Which algebraic/operator language (spin, fermion, bosons...)?

The choice of preferred subsystems becomes problematic
in the presence of nontrivial physical or operational constraints.

Desiderata for a generalized theory:

- ➞ Consistent with existing theory/results in well-characterized limits
- ➞ Directly applicable to arbitrary many-body systems and operator languages
- ➞ Flexible in incorporating physical constraints

A possible candidate:
Generalized Entanglement...
(GE)

The notion of GE

Keyword: Define GE relative to a distinguished subspace of observables.

[Barnum *et al*, PRA 2003; PRL 2004]

- Steps toward GE:

(1) Recall that pure entangled states are those for which at least one subsystem-state is mixed.

(2) Consider states as positive linear functionals on operators:

$$\mathcal{H}\text{-state } |\psi\rangle: \quad \lambda : \mathrm{End}(\mathcal{H}) \rightarrow \mathrm{R}, \quad \lambda(\mathrm{X}) = \mathrm{Tr}(|\psi\rangle\langle\psi|\mathrm{X}) = \langle\psi|X|\psi\rangle$$

A reduced state relative to Ω is defined only by expectation values of observables in Ω:

$$\Omega\text{-state:} \quad \omega : \Omega \rightarrow \mathrm{R}, \quad \omega = \lambda \,|\, \Omega$$

(3) Observe that the set of Ω-reduced states is convex:

$$x, y \in C \;\Rightarrow\; px + (1-p)\,y \in C, \quad p \in [0,1]$$

An Ω-reduced state is pure iff it is extremal i.e., it cannot be written as a convex combination of other reduced states.

Degree of entanglement directly determined by expectations of physical observables:

A pure state is generalized unentangled relative to Ω if its reduced state is pure (extremal), generalized entangled otherwise.

→ Standard extension to mixed states:

A mixed state is generalized unentangled relative to Ω if it is a mixture of generalized unentangled pure states.

Focus on pure states here...

The Lie-algebraic GE setting

Keyword: Ω is a (semisimple) Lie algebra $\mathfrak{h}$, irreducibly represented in $\mathcal{H}$.

- Natural GE measure: Let $\{x_i\}$ be a Hermitian, orthogonal basis for $\mathfrak{h}$. Define $\mathfrak{h}$-purity by

$$P_{\mathfrak{h}}(|\psi\rangle) = K \sum_i \left|\langle\psi|X_i|\psi\rangle\right|^2$$

K is a global normalization factor chosen such that $P_{\mathfrak{h}}^{\max} = 1$ for all generalized unentangled $|\psi\rangle$.

→ Geometrical meaning:

$P_{\mathfrak{h}}(|\psi\rangle) = Tr\left((\Pi_{\mathfrak{h}}|\psi\rangle\langle\psi|)^2\right) =$ Square length of projection of $|\psi\rangle\langle\psi|$ onto $\mathfrak{h}$.

→ Invariance under group transformations: $P_{\mathfrak{h}}(|\psi\rangle) = P_{\mathfrak{h}}(D|\psi\rangle)$, $D = \exp(i\sum_l \eta_l X_l) \in G$, $\eta_l \in \mathbb{R}$

- Complete characterization of set of generalized-unentangled states:

A pure state is generalized unentangled relative to $\mathfrak{h}$ iff it is a Generalized Coherent State (GCS) of the Lie group generated by $\mathfrak{h}$.

$$|GCS(\vec{\alpha})\rangle = \exp\left(\sum_k \alpha_k A_k - \alpha_k^* A_{-k}\right)|REF\rangle, \quad \alpha_l \in \mathbb{C}$$

→ GCSs have max $\mathfrak{h}$-purity;

→ GCSs have min invariant uncertainty... $(\Delta I)^2 = \sum_i \left[\langle X_i^2\rangle - \langle X_i\rangle^2\right] = \langle C_2\rangle - P_{\mathfrak{h}}$

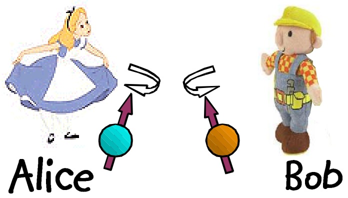
Most classical states...

Example I: Standard entanglement revisited

Bipartite setting: $\mathcal{H} \simeq \mathcal{H}_A \otimes \mathcal{H}_B$ $\quad \dim \mathcal{H}_A = m,\ \dim \mathcal{H}_B = n$

→ Means for manipulating/observing systems are restricted to arbitrary local observables:

$$\mathfrak{h}_{loc} = \mathfrak{h}_A \oplus \mathfrak{h}_B = \{ A \otimes \mathbf{I} + \mathbf{I} \otimes B \} = \mathfrak{su}(m) \oplus \mathfrak{su}(n)$$



→ GCSs of SU(m) x SU(n) are all states reachable from $|0\rangle_A \otimes |0\rangle_B$ via local unitary transformations...

Natural generalization:

Multipartite setting:

Standard multipartite entanglement $\equiv$ GE relative to all local observables

Special case: N spin-1/2 particles

→ Local spin observables are distinguished: $\mathfrak{h}_{loc} = \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \ldots \oplus \mathfrak{su}(2)_N = \text{span}\{\sigma_\alpha^i \mid \alpha = x, y, z \}$

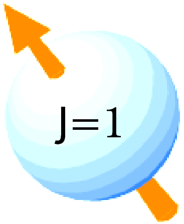
$$P_{loc}(|\psi\rangle) = \frac{1}{N} \sum_{i,\alpha} \langle \psi | \sigma_\alpha^i | \psi \rangle^2 = \frac{2}{N} \sum_i Tr\, \rho_i^2 - 1$$

- The local purity is proportional to the average subsystem purity (global entanglement).

[Meyer & Wallach, JMP 2002]

- Different choices of algebras can probe different aspects of quantum correlations.

Example II: GE without subsystems



System: A single spin-1 particle

→ State space $\mathcal{H} \simeq \mathbb{C}^3$:

- Carries the spin-1 irrep of $su(2) = \text{span}\{J_x, J_y, J_z\}$
- $\mathfrak{h} = \{\text{CSA} \oplus \mathfrak{h}_+ \oplus \mathfrak{h}\_\}$, $\text{CSA} = \text{span}\{J_z\}$, $\mathfrak{h}_+ = \text{span}\{J_+\}$, $\mathfrak{h}_- = \text{span}\{J_-\}$
- $|REF\rangle = |j=1, m=1\rangle$ is the highest-weight reference state.

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

→ Assume that distinguished observables are linear in angular momentum: $\mathfrak{h} = su(2)$

- The reduced states may be identified with vectors of expectations of the generators:

$$\lambda_{\text{red}} \Leftrightarrow \left(\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle\right) \in \mathbb{R}^3, \quad \text{with} \quad \langle J_x\rangle^2 + \langle J_y\rangle^2 + \langle J_z\rangle^2 \leq 1$$

- Pure states are those on the surface = SU(2) angular momentum spin coherent states:

$$(\boldsymbol{n}\cdot\boldsymbol{J})|\xi\rangle = \pm|\xi\rangle, \quad |\xi\rangle = \exp\left(\xi J_+ - \xi^* J_-\right)|1,-1\rangle, \quad \xi \in \mathbb{C}$$

- $|1,-1\rangle, |1,1\rangle$ are GCSs, $|1,0\rangle$ is not: $|1,0\rangle$ is generalized entangled relative to $su(2)$.

$$|1,0\rangle \approx \frac{1}{2}(|1,1\rangle\langle 1,1| + |1,-1\rangle\langle 1,-1|)$$

Indistinguishable from mixture based on SU(2)-expectations...

→ All pure states are unentangled relative to $\mathfrak{h} = su(3)$.

Example III: Fermionic GE

System: N spinless fermion modes e.g. spatial sites, momentum modes…

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \quad \{c_i, c_j\} = 0, \quad \{c_i^\dagger, c_j^\dagger\} = 0$$

➞ Associate “local” resources with number-preserving fermionic operators:

$$\mathfrak{h} = \mathfrak{u}(N) = \text{span}\left\{ c_i^\dagger c_i - \frac{1}{2}, \frac{c_i^\dagger c_j + c_j^\dagger c_i}{\sqrt{2}}, \frac{c_i^\dagger c_j - c_j^\dagger c_i}{i\sqrt{2}} \right\} \qquad 1 \le i < j \le N$$

$$P_{\mathfrak{u}(N)}(|\psi\rangle) = \frac{2}{N}\sum_{j<j'=1}^{N}\left[\langle c_j^\dagger c_{j'} + c_{j'}^\dagger c_j\rangle^2 - \langle c_j^\dagger c_{j'} - c_{j'}^\dagger c_j\rangle^2\right] + \frac{4}{N}\sum_{j=1}^{N}\left\langle c_j^\dagger c_j - \frac{1}{2}\right\rangle^2$$

- The GCSs of $\mathfrak{u}(N)$ are the fermionic product states = Slater determinants

$$|GCS(N)\rangle = \prod_l c_l^\dagger |VAC\rangle$$

- The fermionic purity $P_{\mathfrak{u}(N)} = 1$ for any Slater determinant (with any number of fermions);

  $P_{\mathfrak{u}(N)} < 1$ for any other (non-extremal) fermionic state e.g., N=2, use Jordan-Wigner mapping:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \longleftrightarrow \frac{1}{\sqrt{2}}\left(c_1^\dagger|VAC\rangle - c_2^\dagger|VAC\rangle\right)$$

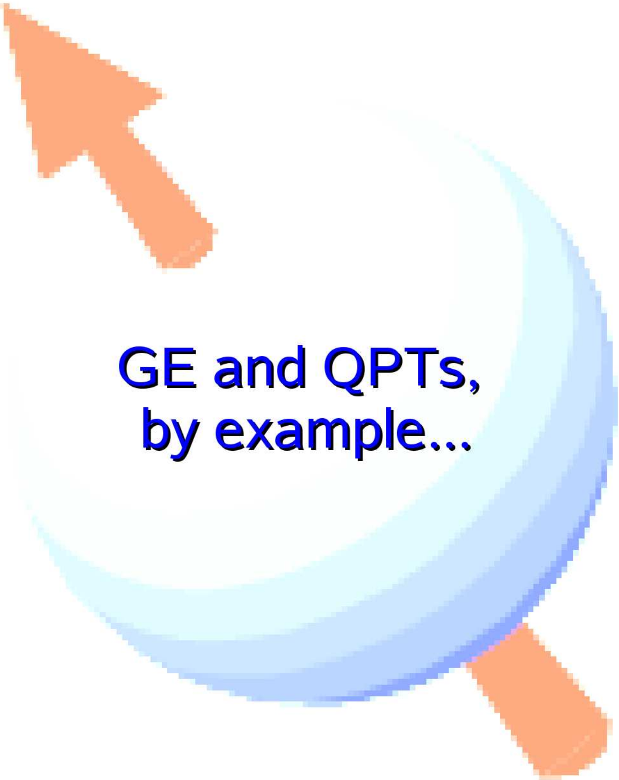
Mode-entangled ($su(2)\oplus su(2)$),
$\mathfrak{u}(N)$-unentangled

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \longleftrightarrow \frac{1}{\sqrt{2}}\left(|VAC\rangle - c_1^\dagger c_2^\dagger|VAC\rangle\right)$$

Max $\mathfrak{u}(N)$-entangled

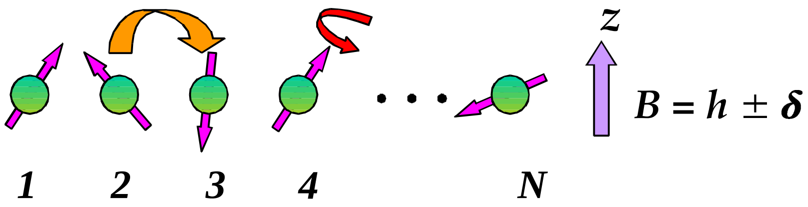
➞ Fermionic GE is independent on both the set of modes and the operator language chosen!

$$c_j \to \sum_m U_{mj}\bar{c}_m, \quad U \in Mat(N \times N).$$



GE and QPTs,
by example...

Case study: Anisotropic XY model in alternating tranverse field



- Hamiltonian for a regularly inhomogeneous spin-1/2 chain (N even, periodic BCs, $\sigma_a^{N+1} = \sigma_a^1$):

$$H = -\sum_{i=1}^{N} \left(\frac{(1+\gamma)}{2} \sigma_x^i \sigma_x^{i+1} + \frac{(1-\gamma)}{2} \sigma_y^i \sigma_y^{i+1} \right) + \sum_{i=1}^{N} \left(h - (-)^i \delta \right) \sigma_z^i$$

$\gamma \in [0, 1]$: anisotropy; $h \in [-\infty, +\infty]$: magnetic field strength; $\delta \in [-\infty, +\infty]$: alternation strength

$\delta = 0$: Anisotropic XY model in transverse magnetic field [Somma *et al*, PRA 2004]

$\delta > 0, \gamma = 1$: Ising model in alternating transverse field [Derzhko *et al*, PRE 2004]

$\gamma = 0$: Isotropic XX limit

- Symmetries:
  - ➞ For generic values of the parameters, H has a global discrete $\mathbb{Z}_2$-symmetry, $\mathbb{Z}_2^z = \Pi_{j=1}^{N} \sigma_z^j$, which is spontaneously broken in the thermodynamic limit.
  - ➞ For specific values of the parameters, H may develop additional symmetries:
    - $\gamma = 0$: Continuous $u(1)$-symmetry under arbitrary z-rotation;
    - h=0: Discrete symmetry under global x-rotation followed by lattice translation:

$$\mathrm{W} = \mathrm{T}\mathbb{Z}_2^{x}, \quad \mathbb{Z}_2^{x} = \Pi_{j=1}^{N} \sigma_x^j, \quad \mathrm{T}: j \rightarrow j+1$$

Exact solution

Steps:

(1) Generalized even-odd Jordan-Wigner transformation:

$$a^\dagger_{2j-1} = \left(\prod_{m=1}^{2j-2}(-\sigma_z^m)\right)\sigma^\dagger_{2j-1}, \quad b^\dagger_{2j} = \left(\prod_{m=1}^{2j-1}(-\sigma_z^m)\right)\sigma^\dagger_{2j}, \quad j=1,\ldots,N/2$$

(2) Fourier-transform to momentum modes:

$$a^\dagger_k = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} \mathrm{e}^{i\,k\,(2j-1)}\, a^\dagger_{2j-1}, \quad b^\dagger_k = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} \mathrm{e}^{i\,k\,(2j)}\, b^\dagger_{2j}, \quad k \in K_+ + K_- = \left\{\pm\frac{\pi}{N}, \pm\frac{3\pi}{N}, \ldots, \pm\left(\frac{\pi}{2}-\frac{\pi}{N}\right)\right\}$$

(3) Block diagonalization/Bogoliubov quasiparticle transformation:

$$H = \sum_{k\in K_+} A^\dagger_k H_k A_k = \sum_{k\in K_+}^{n=1,\ldots,4} \epsilon_{k,n}\, \gamma^\dagger_{k,n}\, \gamma_{k,n}$$

$$A_k = \begin{pmatrix} a_k \\ a^\dagger_{-k} \\ b_k \\ b^\dagger_{-k} \end{pmatrix}, \quad H_k = \begin{pmatrix} 2(h+\delta) & 0 & J_k & \Gamma_k \\ 0 & -2(h+\delta) & -\Gamma_k & J_k \\ \bar{J}_k & -\bar{\Gamma}_k & 2(h-\delta) & 0 \\ \bar{\Gamma}_k & \bar{J}_k & 0 & -2(h-\delta) \end{pmatrix}, \quad \begin{array}{l} J_k = -2\cos(k), \\ \Gamma_k = -2i\gamma\sin(k) \end{array}$$

➡ Zero temperature ground-state energy and ground-state structure:

$$E_{GS} = \sum_{k\in K_+}\left(\epsilon_{k,1}+\epsilon_{k,2}\right), \qquad \epsilon_{k,1}<0,\ \epsilon_{k,2}\leqslant 0$$

$$|GS\rangle = \prod_{k>0}\left(u^1_k + u^2_k\, a^\dagger_k\, a^\dagger_{-k} + u^3_k\, b^\dagger_k\, b^\dagger_{-k} + u^4_k\, a^\dagger_k\, b^\dagger_{-k} + u^5_k\, a^\dagger_{-k}\, b^\dagger_k + u^6_k\, a^\dagger_k\, a^\dagger_{-k}\, b^\dagger_k\, b^\dagger_{-k}\right)|VAC\rangle$$

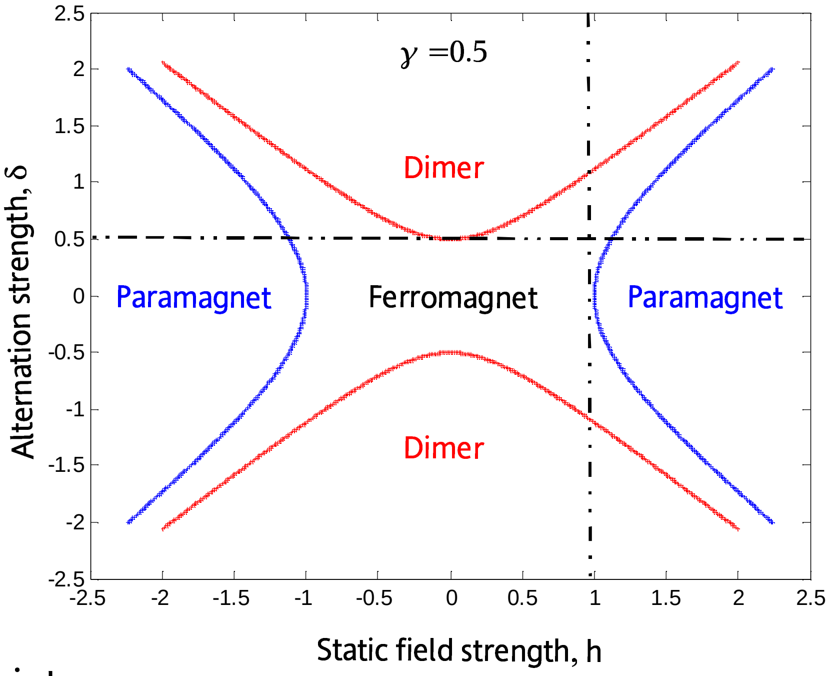
Static quantum criticality properties

- Quantum phases:

  QCPs $(h_c, \delta_c, \gamma_c)$ are determined by zeroes of $\epsilon_{k,2}$.

  Quantum phase boundaries:

$$h^2 = \delta^2 + 1$$
$$\delta^2 = h^2 + \gamma^2$$



➞ PM/FM phase boundary is characterized by 2^{nd} order broken-symmetry QPT;

➞ Ground state develops weak singularities at

$$(h_c, \delta_c, \gamma_c) = (0, \delta = \pm\gamma)$$
$$(h_c, \delta_c, \gamma_c) = (\pm 1, \delta = 0)$$

4^{th} order broken-symmetry QPTs occur at these points.

➞ In the isotropic limit, an insulator-metal Lifshitz QPT occurs, with no symmetry order parameter.

Focus on broken-symmetry QPTs...

- Universality classes:

  Standard finite-size scaling analysis reveals the emergence of new quantum critical behavior in the presence of alternation:

Ising universality class:	$\nu = 1\,,\ z = 1$
Alternating universality class:	$\nu = 2\,,\ z = 1$

GE as a QPT indicator: Phase diagram

- Relevant (Lie) algebras of observables acting on the 2^N-dimensional spin space:

  $u(N)$= {number-conserving quadratic fermionic operators } $\subset so(2N)$

  ➞ The GS is always a GCS of $so(2N)$, GE relative to $so(2N)$ carries no information about QCPs.

  ➞ The GS becomes a GCS of $u(N)$ in the fully polarized PM limit...

$$P_{u\,(N)}(|GS\,\rangle)=\frac{4}{N}\sum_k \langle a_k^\dagger a_k - 1/2\rangle^2 + \langle a_{-k}^\dagger a_{-k} - 1/2\rangle^2 + \langle b_k^\dagger b_k - 1/2\rangle^2 + \langle b_{-k}^\dagger b_{-k} - 1/2\rangle^2 +$$
$$+2|\langle a_k^\dagger b_k\rangle|^2 + 2|\langle a_{-k}^\dagger b_{-k}\rangle|^2$$

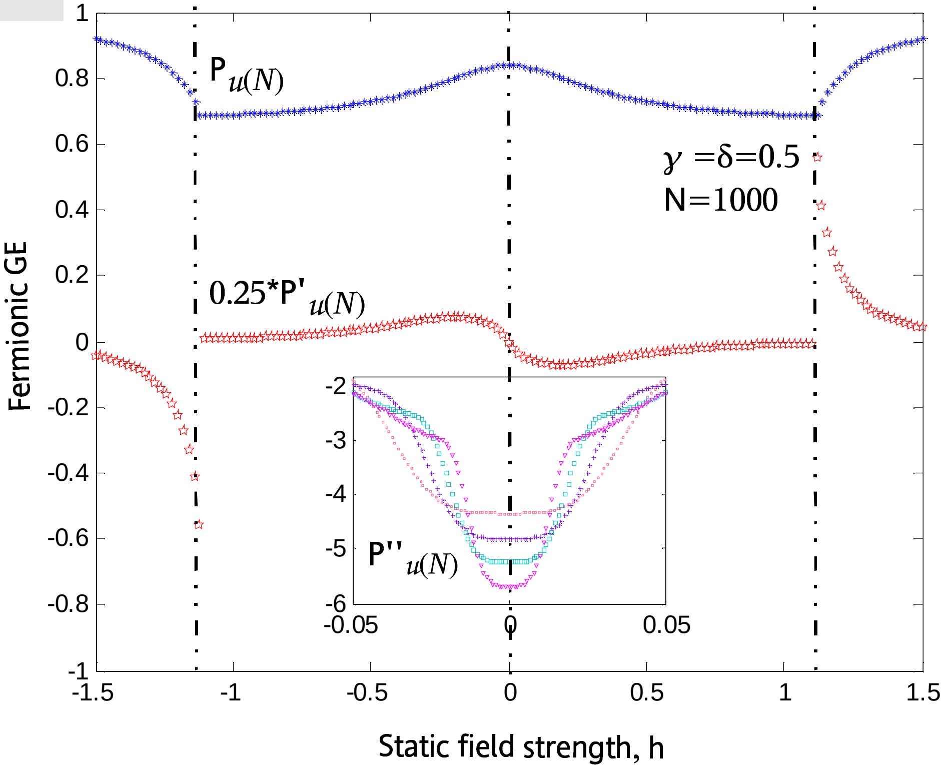
- Ground-state fermionic GE faithfullly portraits underlying quantum phase:

  ➞ Analytical result available for $\delta = 0$;

  ➞ GE sharply detects PM-FM QPTs;

  [Somma *et al*, PRA 2004]

  ➞ Derivatives of GE develop singular behavior (only) at QCPs.



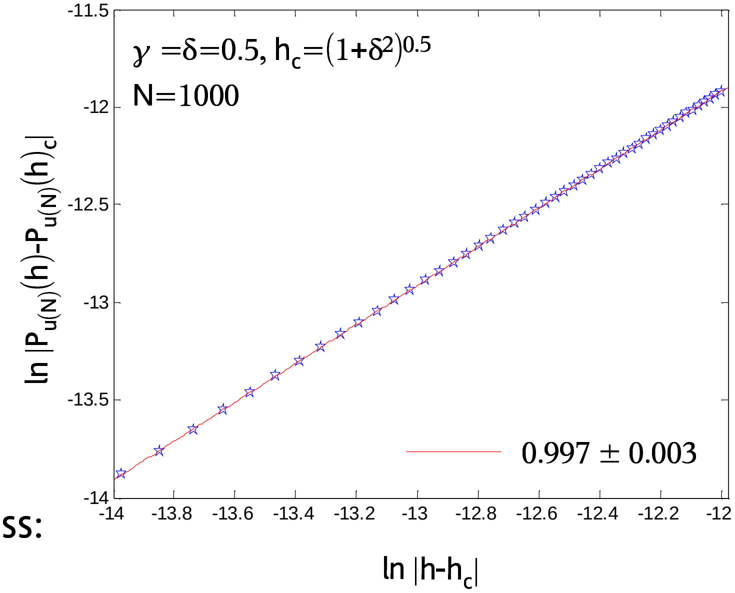
GE as a QPT indicator: Scaling properties

- Ground-state fermionic GE contains complete information about static critical exponents:
  - ➞ Taylor-expand purity near QCP;

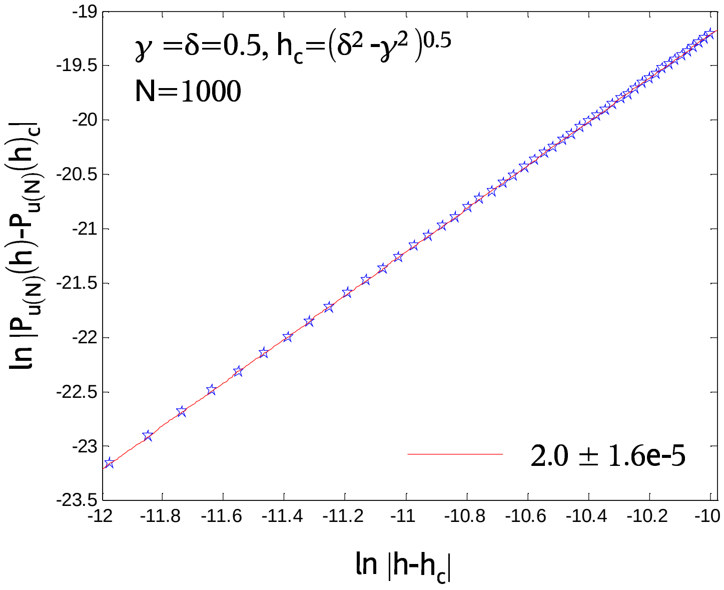
Hint…

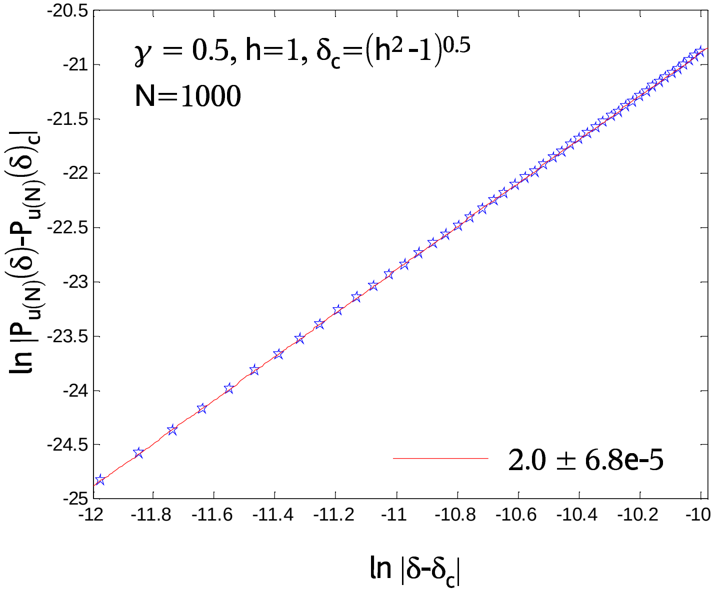
- ➞ Ground-state fermionic GE is related to the fluctuations of the total fermionic number operator…

Ising universality class:



Alternating universality class:





Dynamic QPTs and the Kibble-Zurek mechanism

Can nonequilibrium properties be predicted using equilibrium critical exponents?

Simplest dynamical scenario: Slow linear sweep of control parameter with constant speed τ_Q

$$g(t)-g_c=\frac{t-t_c}{\tau_Q}, \quad \tau_Q>0,\, t_c=0$$

➞ System response determined by relaxation time

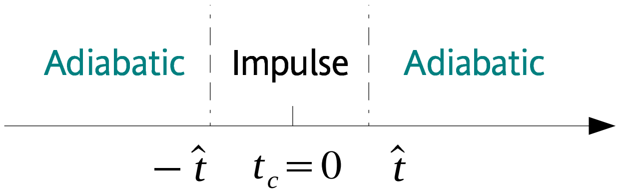
$$\tau=\frac{\hbar}{\Delta}\sim\frac{1}{|g(t)-g_c|^{z\nu}}$$

Δ = Gap between ground and first accessible excited state

Divergent in the thermodynamic limit for arbitrarily slow quenches: Critical slowing-down

KZM: Crossover from (approximately) adiabatic to impulse regime at freeze-out time

$$\tau(\hat{t})=\left|\frac{g(t)-g_c}{g'(\hat{t})}\right| \Rightarrow \boxed{\hat{t}\sim\tau_Q^{\nu z/(\nu z+1)}}$$



➞ Prediction for scaling of final density of excitations:

$$\boxed{n(t_F)\sim\tau_Q^{-\nu/(\nu z+1)}}$$

[Zurek, Dorner, Zoller, PRL 2005; Dziarmaga, PRL 2005...]

Still...

(1) What is the nature of the KZM? Does it apply only to 2nd order QPTs?...
(2) What features of the initial/final quantum phase are relevant?...
(3) How does dynamical scaling reflect into entanglement and/or observable properties?...

Non-equilibrium excitation density

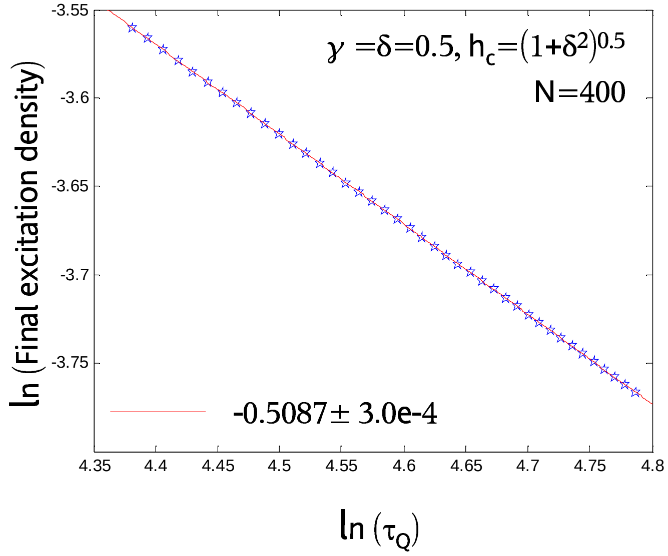
$$|\Psi(t)\rangle = \prod_k \left(u_k^1(t) + u_k^2(t)\, a_k^\dagger a_{-k}^\dagger + u_k^3(t)\, b_k^\dagger b_{-k}^\dagger + u_k^4(t)\, a_k^\dagger b_{-k}^\dagger + u_k^5(t)\, a_{-k}^\dagger b_k^\dagger + u_k^6(t)\, a_k^\dagger a_{-k}^\dagger b_k^\dagger b_{-k}^\dagger\right)|VAC\rangle$$

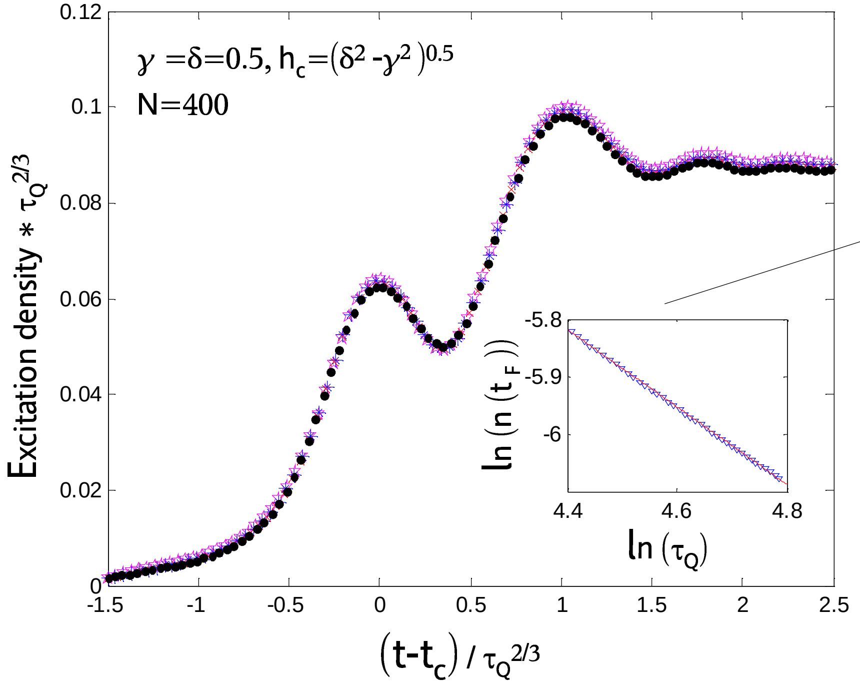
- Final excitation density:

$$n(t_F) = \frac{1}{N}\langle \Psi(t_F)|\sum_k \left(\gamma_{k,3}^\dagger \gamma_{k,3} + \gamma_{k,4}^\dagger \gamma_{k,4}\right)|\Psi(t_F)\rangle$$

➞ Agrees with KZM prediction over appropriate τ_Q-range irrespective of details of the QCP/quantum phase:

$$n(t_F)^{Ising} \sim \tau_Q^{-1/2}, \quad n(t_F)^{Alt} \sim \tau_Q^{-2/3}$$





Inset: —— -0.672 ± 0.006

- Time-dependent excitation density:

$$n(t) = \tau_Q^{-\nu/(\nu z+1)} F\left(\frac{t-t_c}{\hat{t}}\right)$$

➞ Scaling behavior holds throughout entire time evolution...

Non-equilibrium GE scaling

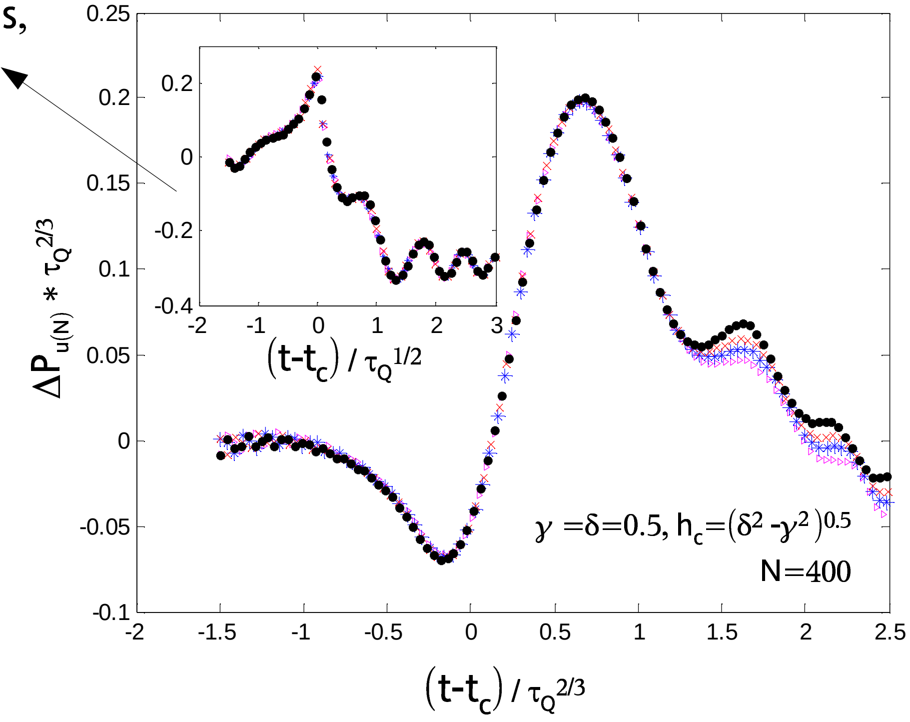
- Fermionic GE also obeys scaling behavior across the entire dynamics provided that the amount relative to the instantaneous static ground state is considered:

$$\Delta P_{u\,(N)}(t) = P_{u\,(N)}(|\Psi(t)\rangle) - P_{u\,(N)}(|\Psi_0(t)\rangle) = \tau_Q^{-\nu/(\nu z+1)} G\left(\frac{t-t_c}{\hat{t}}\right)$$

Exact
time-evolved state

Instantaneous
ground state

$$|\Psi_0(t)\rangle \equiv |GS(t)\rangle$$



Inset: Ising class,
$\gamma = \delta = 0.5$,
$h_c = (1+\delta^2)^{0.5}$

Toward dynamical quantum critical scaling...

Numerical evidence: Arbitrary physical observables obey scaling behavior of the form

$$\Delta O(t) = O(|\Psi(t)\rangle) - O(|\Psi_0(t)\rangle) = \tau_Q^{-\nu(d+xz)/(\nu z+1)} F\left(\frac{t-t_c}{\hat{t}}\right)$$

for some scaling function F and factor x depending on O, its dimension, and the control path, e.g. d=1

$$\Delta H(t) = H(|\Psi(t)\rangle) - H(|\Psi_0(t)\rangle) = \tau_Q^{-\nu(1+z)/(\nu z+1)} f\left(\frac{t-t_c}{\hat{t}}\right)$$

Why?

Hint: For gapped systems, effect of adiabaticity-breaking at criticality can still be accounted for by a perturbative argument...

[Polkovnikov, PRB 2005; cmat/0706.0212]

$$H(t) = H_0 + [(g(t) - g_c)+g_c]H_1 = H(g=g_c) + t/\tau_Q\, H_1\,, \qquad H(t)|\Psi(t)\rangle = E_n(t)\Psi_n(t)$$

(1) Represent time-evolved state using adiabatic perturbation theory around snapshot eigenstates:

$$|\Psi(t)\rangle = e^{i\,\Gamma(t)}\left\{|\Psi_0(t)\rangle - i\,g'\sum_{n\neq 0}\frac{\langle\Psi_n(t)|H_1|\Psi_0(t)\rangle}{(E_n(t)-E_0(t))^2}|\Psi_n(t)\rangle + O(g'^2)\right\}$$

(2) Supplement description with scaling assumptions:

$$E_n(t)-E_0(t) = (t/\tau_Q)^{z\nu} F\left(\frac{\Delta_n}{(t/\tau_Q)^{z\nu}}\right), \qquad \langle\Psi_n(t)|H_1|\Psi_0(t)\rangle = (t/\tau_Q)^{z\nu-1} G\left(\frac{\Delta_n}{(t/\tau_Q)^{z\nu}}\right),$$

$$\rho(E) \propto E^{d/z-1}\ldots$$

[Deng, Ortiz, Viola, in progress]

Conclusion and outlook

Entanglement is – inevitably – a relative concept...

- GE provides a unifying framework for defining entanglement relative to any physically relevant, distinguished subspace of observables.
- GE is directly applicable to both distinguishable and indistinguishable degrees of freedom and relates naturally to generalized coherent state theory.
- GE provides useful diagnostic tools for "complex" quantum systems – in particular, quantum critical systems at equilibrium and beyond.

Are we capturing the right relativity? Only time will tell...
Meanwhile...

(1) GE and QPTs:

- ✓ Static QPTs: Validate analysis on different models/algebras?...
- ✓ Dynamical QPTs: Continue/extend analysis and develop general framework?...
- ✓ Can GE detect criticality signatures in excited states?...

(2) GE and quantum chaos:

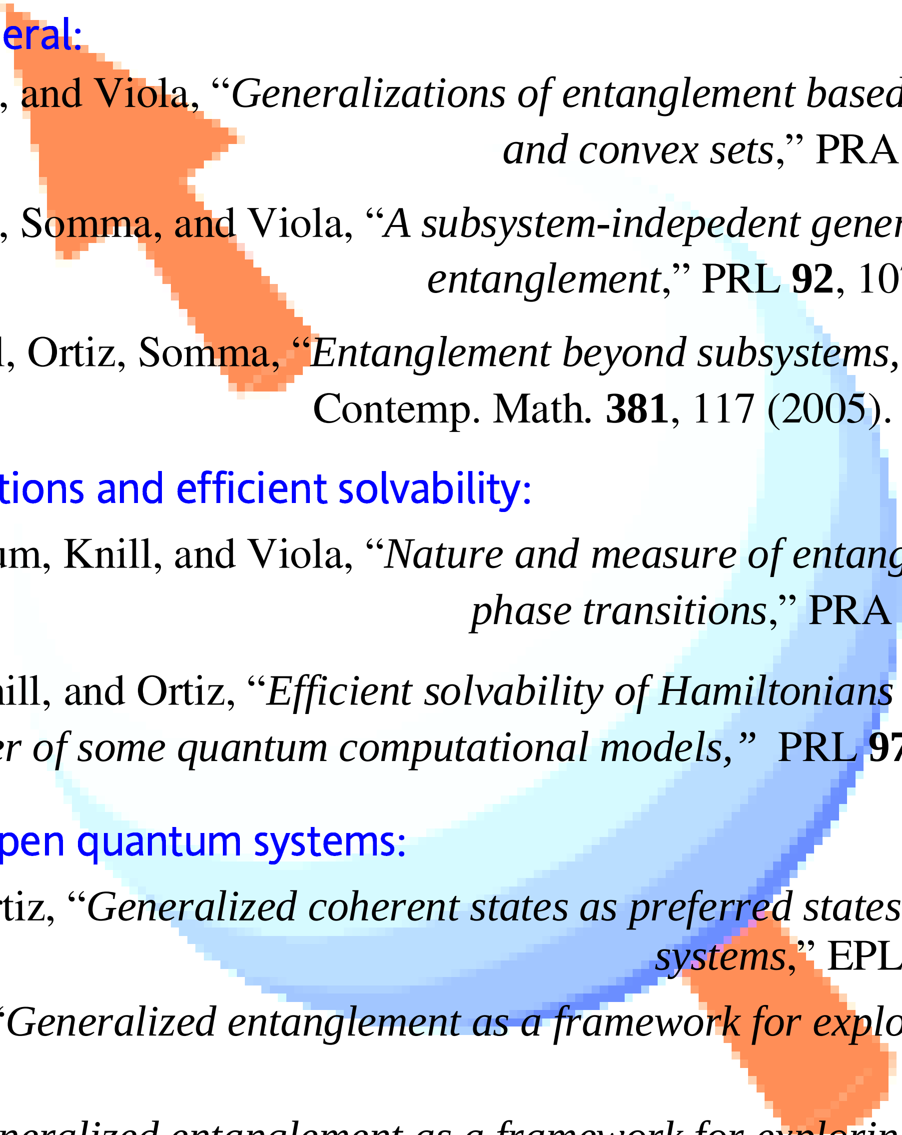
- ✓ Can GE suggest reliable indicators for different integrability regimes?...

(3) GE and open quantum systems:

- ✓ What determines stability properties of GE under open-system dynamics?...
- ✓ Estimation-based characterization of GE and "GE-assisted metrology"?...

Thank you for your attention!

Further reading on GE...



(1) Mathematical and general:

- ✔ Barnum, Knill, Ortiz, and Viola, "*Generalizations of entanglement based on coherent states and convex sets*," PRA **68**, 032308 (2003).
- ✔ Barnum, Knill, Ortiz, Somma, and Viola, "*A subsystem-indepedent generalizations of entanglement*," PRL **92**, 107902 (2004).
- ✔ Viola, Barnum, Knill, Ortiz, Somma, "*Entanglement beyond subsystems,*" Contemp. Math. **381**, 117 (2005).

(2) Quantum phase transitions and efficient solvability:

- ✔ Somma, Ortiz, Barnum, Knill, and Viola, "*Nature and measure of entanglement in quantum phase transitions*," PRA **70**, 042311 (2004).
- ✔ Somma, Barnum, Knill, and Ortiz, "*Efficient solvability of Hamiltonians and limits on the power of some quantum computational models,*" PRL **97**, 190501 (2006).

(3) Quantum chaos and open quantum systems:

- ✔ Boixo, Viola, and Ortiz, "*Generalized coherent states as preferred states of open quantum systems*," EPL, in press (2007).
- ✔ Weinstein & Viola, "*Generalized entanglement as a framework for exploring quantum chaos,*" EPL **76**, 746 (2006).
- ✔ Viola & Brown, "*Generalized entanglement as a framework for exploring complex quantum systems: Purity vs delocalization measures,*"JPA **40**, 8109 (2007).