THERMAL AND MORE GENERAL MIXED COHERENT STATES FOR OPEN SYSTEMS AND QUANTUM INFORMATION THEORY

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OUTLINE

- Single-boson (Glauber) pure coherent states
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- Displaced negative-binomial mixed coherent states
- Generalization of thermofield dynamics

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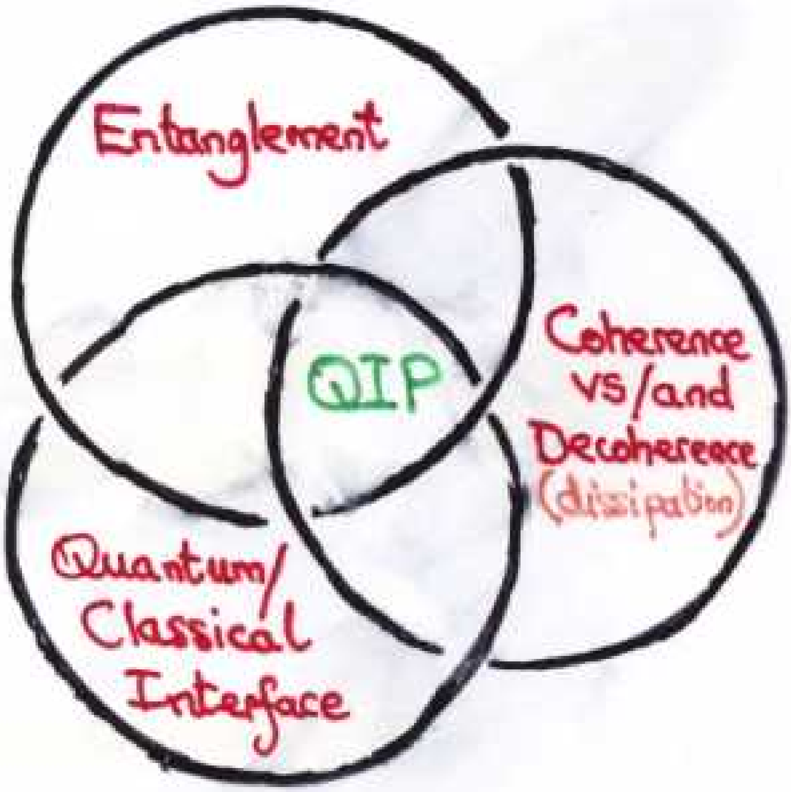
The Borromean Rings of Quantum Information Processing



[– "strength in unity"
– "united we stand, divided we fall"
– …]

Q: Are there any theoretical formalisms that address simultaneously (and naturally) all three aspects?

Single-boson (Glauber) pure coherent states

- Consider a single-mode harmonic oscillator

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \quad ; \qquad [\hat{x},\hat{p}] = i\hbar\mathbb{1}$$

$$\equiv \hbar\omega\left(\hat{a}_0^\dagger\hat{a}_0 + \tfrac{1}{2}\right) \equiv \hbar\omega\left(\hat{n}_0 + \tfrac{1}{2}\right) ; \quad \hat{n}_0 \equiv \hat{a}_0^\dagger\hat{a}_0$$

with $\hat{a}_0 \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \Rightarrow [\hat{a}_0, \hat{a}_0^\dagger] = \mathbb{1}$

- number eigenstates : $\hat{n}_0|n\rangle = n|n\rangle$

$$\Rightarrow \quad |n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}_0^\dagger)^n|0\rangle \; ; \quad n = 0,1,2,\ldots$$

– vacuum state : $\hat{a}_0|0\rangle = 0 \; ; \; \langle 0|0\rangle = 1$

– orthonormality : $\langle m|n\rangle = \delta_{mn}$

– completeness : $\sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbb{1}$

– also : $\hat{a}_0|n\rangle = \sqrt{n}\,|n-1\rangle$
$\hat{a}_0^\dagger|n\rangle = \sqrt{(n+1)}\,|n+1\rangle$

- $|z\rangle \equiv |x,p\rangle$ is a Glauber coherent state if

$$\hat{a}_0|z\rangle = z|z\rangle \; ; \text{ right eigenstate of } \hat{a}_0$$

– put $z \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right) \; ; \; x,p \in \mathbb{R}$

$$\Leftrightarrow \quad \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)|x,p\rangle = \left(x + \frac{ip}{m\omega}\right)|x,p\rangle$$

– expanding $|z\rangle$ in terms of number eigenstates

$$\Rightarrow \quad |z\rangle = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle = e^{-\frac{1}{2}|z|^2}e^{za_0^\dagger}|0\rangle$$

– alternatively, using the disentangling relation for the Heisenberg-Weyl group (of operators $\hat{a}_0, \hat{a}_0^\dagger, \mathbb{1}$):

$$e^{(z\hat{a}_0^\dagger - z^*\hat{a}_0)} = e^{-\frac{1}{2}|z|^2}e^{z\hat{a}_0^\dagger}e^{-z^*\hat{a}_0}$$

we can write $|z\rangle$ in terms of

- the unitary Weyl displacement operator :

$$\boxed{\hat{D}(z) \equiv e^{z\hat{a}_0^\dagger - z^*\hat{a}_0}} \qquad [\hat{D}(z)]^\dagger = [\hat{D}(z)]^{-1} = \hat{D}(-z)$$

as $$\boxed{|z\rangle = \hat{D}(z)|0\rangle}$$

$$\Leftrightarrow \boxed{\hat{D}(x,p) \equiv e^{\frac{i}{\hbar}(p\hat{x} - x\hat{p})}} \quad [\hat{D}(x,p)]^\dagger = [\hat{D}(x,p)]^{-1} = \hat{D}(-x,-p)$$

$$\boxed{|x,p\rangle = \hat{D}(x,p)|0\rangle}$$

- We may readily check that :

$$\hat{D}(z)\,\hat{a}_0\,\hat{D}^\dagger(z) = \hat{a}_0 - z \equiv \hat{a}_z$$

$$\hat{D}(z)\,\hat{a}_0^\dagger\,\hat{D}^\dagger(z) = \hat{a}_0^\dagger - z^* \equiv \hat{a}_z^\dagger$$

$$\hat{D}(x_0,p_0)\,\hat{x}\,\hat{D}^\dagger(x_0,p_0) = \hat{x} - x_0$$

$$\hat{D}(x_0,p_0)\,\hat{p}\,\hat{D}^\dagger(x_0,p_0) = \hat{p} - p_0$$

* * *

- The Glauber coherent states are massively overcomplete, but they do provide a
- resolution of the identity operator

$$\boxed{\begin{array}{c} \int \frac{d^2z}{\pi}\,|z\rangle\langle z| = \mathbb{1} \\ d^2z \equiv dz_R dz_I = \frac{dz\,dz^*}{2i} \end{array}}$$

integration over the entire complex z-plane

$$\Leftrightarrow \boxed{\iint \frac{dx\,dp}{2\pi\hbar}\,|x,p\rangle\langle x,p| = \mathbb{1}}$$

integration over whole classical phase space
$-\infty < x < \infty;\ -\infty < p < \infty$

* * *

- More general coherent states $|\ell\rangle$ are required only to satisfy the following 2 conditions :

▲ (1) Continuity : the vector $|\ell\rangle$ is a strongly continuous function of the label ℓ $(\ell \in \mathcal{L})$

[Notes: (a) continuity has its usual meaning, viz., as $\ell' \to \ell$, $\| |\ell'\rangle - |\ell\rangle \| \to 0$ where $\| |\psi\rangle \| \equiv \langle\psi|\psi\rangle^{1/2}$, as usual. We suppose $|\ell\rangle \neq 0 \ \forall \ell \in \mathcal{L}$

(b) Clearly, such vectors $|\ell\rangle$ will never be mutually orthogonal]

▲ (2) resolution of the identity ("completeness")

There exists a positive measure $\delta\ell$ on $\mathcal{L}$ such that

$$\boxed{\mathbb{1} = \int_{\mathcal{L}} \delta\ell \, |\ell\rangle\langle\ell|} \qquad (1)$$

* * *

Note: Eq. (1) $\Rightarrow$

$$\langle\ell|\psi\rangle = \int_{\mathcal{L}} \delta\ell \, \langle\ell|\ell'\rangle\langle\ell'|\psi\rangle$$

and hence each admissible function $\langle\ell|\psi\rangle$ satisfies the above homogeneous integral equation involving the reproducing kernel

$$\mathcal{K}(\ell,\ell') \equiv \langle\ell|\ell'\rangle = \langle\ell'|\ell\rangle^*$$

c.f., for Glauber coherent states:

$$\boxed{\mathcal{K}(z,z') \equiv \langle z|z'\rangle = \exp\left(-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|z'|^2 + z^* z'\right)}$$

* * *

• P-, Q-, and W-representations

- It is well known in quantum optics (Klauder et al.) that all operators belonging to the trace class (i.e., all $\hat{\Theta}$ for which $\mathrm{Tr}\,\hat{\Theta} < \infty$) are uniquely specified by either their Q-representation or their P-representation:

$$\boxed{\begin{array}{l} Q(\hat{\Theta};z) \equiv \langle z|\hat{\Theta}|z\rangle \Leftrightarrow Q(\hat{\Theta};x,p) \equiv \langle x,p|\hat{\Theta}|x,p\rangle \\ \hline \hat{\Theta} = \int \frac{d^2z}{\pi} P(\hat{\Theta};z)|z\rangle\langle z| \Leftrightarrow \hat{\Theta} = \iint \frac{dx\,dp}{2\pi\hbar} P(\hat{\Theta};x,p)|x,p\rangle\langle x,p| \end{array}}$$

$$\Rightarrow Q(\hat{\Theta};z) = \int \frac{d^2z'}{\pi} P(\hat{\Theta};z')\, e^{-|z-z'|^2} = \exp\left(\tfrac{1}{4}\Delta_z\right) P(\hat{\Theta};z) \quad (2)$$

in terms of the 2D Laplacian operator:

$$\Delta_z \equiv 4\frac{\partial^2}{\partial z\,\partial z^*} = \frac{\partial^2}{\partial z_R^2} + \frac{\partial^2}{\partial z_I^2} = 2\hbar\left(\frac{1}{m\omega}\frac{\partial^2}{\partial x^2} + m\omega\frac{\partial^2}{\partial p^2}\right)$$

- Introduce Fourier transforms:

$$\tilde{f}(w) \equiv \int \frac{d^2z}{\pi} e^{iz\cdot w} f(z) \Leftrightarrow f(z) = \int \frac{d^2w}{4\pi} e^{-iz\cdot w} \tilde{f}(w)$$

where $z\cdot w \equiv \frac{1}{2}(zw^* + z^*w) = z_R w_R + z_I w_I$

Alternatively: $z_R \equiv \sqrt{\frac{m\omega}{2\hbar}}\,x$; $z_I \equiv \sqrt{\frac{1}{2\hbar m\omega}}\,p$, as before,

and $w_R \equiv -\sqrt{\frac{2}{\hbar m\omega}}\,\underline{P}$; $w_I \equiv \sqrt{\frac{2m\omega}{\hbar}}\,\underline{X}$

$$\Rightarrow \quad \tilde{f}(\underline{X},\underline{P}) \equiv \iint \frac{dx\,dp}{2\pi\hbar} e^{i(p\underline{X} - x\underline{P})/\hbar} f(x,p)$$

$$\Leftrightarrow \quad f(x,p) = \iint \frac{d\underline{X}\,d\underline{P}}{2\pi\hbar} e^{i(x\underline{P} - p\underline{X})/\hbar} \tilde{f}(\underline{X},\underline{P})$$

Then, the above relation (2) $\Rightarrow$

$$\boxed{\tilde{Q}(\hat{\Theta};\omega) = e^{-\frac{1}{4}|\omega|^2}\,\tilde{P}(\hat{\Theta};\omega)}$$

- Wigner (W-) and Weyl ($\tilde{W}$-) representations

- Define the Weyl representation of $\hat{\Theta}$ as:

$$\tilde{W}(\hat{\Theta},\omega) \equiv \mathrm{Tr}\left[\hat{\Theta}\hat{D}(\tfrac{1}{2}i\omega)\right]$$

- Its Fourier transform

$$W(\hat{\Theta};z) = \int \frac{d^2\omega}{4\pi}\, e^{-iz\cdot\omega}\,\tilde{W}(\hat{\Theta};\omega)$$

is the Wigner representative of $\hat{\Theta}$

- In terms of position eigenstates $|x\rangle_x$: $\hat{x}|x\rangle_x = x|x\rangle_x$ we readily show (in units where $m=\omega=\hbar=1$)

$$W(\hat{\Theta},z) = \int_{-\infty}^{\infty} dX\; {}_x\langle\sqrt{2}z_R+\tfrac{1}{2}X|\hat{\Theta}|\sqrt{2}z_R-\tfrac{1}{2}X\rangle_x\, e^{-i\sqrt{2}z_I X}$$

$$\tilde{W}(\hat{\Theta},\omega) = \int_{-\infty}^{\infty} dx\; {}_x\langle x+\tfrac{1}{2}\tfrac{\omega_I}{\sqrt{2}}|\hat{\Theta}|x-\tfrac{1}{2}\tfrac{\omega_I}{\sqrt{2}}\rangle_x\, e^{ix\omega_R/\sqrt{2}}$$

$$\Updownarrow$$

$$W(\hat{\Theta};x,p) = \int_{-\infty}^{\infty} dX\; {}_x\langle x+\tfrac{1}{2}X|\hat{\Theta}|x-\tfrac{1}{2}X\rangle_x\, e^{-ipX/\hbar}$$

$$\tilde{W}(\hat{\Theta};X,P) = \int_{-\infty}^{\infty} dx\; {}_x\langle x+\tfrac{1}{2}X|\hat{\Theta}|x-\tfrac{1}{2}X\rangle_x\, e^{-iPx/\hbar}$$

—‖—

- We then rather easily find:

$$\boxed{\begin{aligned} Q(\hat{\Theta};z) &= e^{+\frac{1}{8}\Delta_z}\,W(\hat{\Theta};z) \\ P(\hat{\Theta};z) &= e^{-\frac{1}{8}\Delta_z}\,W(\hat{\Theta},z) \end{aligned}} \Leftrightarrow \boxed{\begin{aligned} \tilde{Q}(\hat{\Theta};\omega) &= e^{-\frac{1}{8}|\omega|^2}\,\tilde{W}(\hat{\Theta};\omega) \\ \tilde{P}(\hat{\Theta};\omega) &= e^{+\frac{1}{8}|\omega|^2}\,\tilde{W}(\hat{\Theta};\omega) \end{aligned}}$$

and hence:

$$\boxed{Q(\hat{\Theta};z) = e^{\frac{1}{4}\Delta_z}\,P(\hat{\Theta};z)} \Leftrightarrow \boxed{\tilde{Q}(\hat{\Theta};\omega) = e^{-\frac{1}{4}|\omega|^2}\,\tilde{P}(\hat{\Theta};\omega)}$$

Mixed thermal coherent states

- Our basic idea is now to use a set of mixed states as a basis in Hilbert space. Previously we used the pure coherent states $|z\rangle$ with projectors

$$\hat{\Pi}(z) \equiv |z\rangle\langle z| \; ; \; \hat{\Pi}^2(z) = \hat{\Pi}(z) \; ; \; \int \frac{d^2z}{\pi} \hat{\Pi}(z) = \mathbb{1}$$

The latter resolution of the identity allowed us to express an arbitrary state $|f\rangle$ as

$$|f\rangle = \int \frac{d^2z}{\pi} f(z) |z\rangle \; ; \; f(z) \equiv \langle z|f\rangle$$

- Now a general mixed state is described by a density operator $\hat{\rho}$ with eigenstates $|e^{(n)}\rangle$ and eigenvalues p_n:

$$\hat{\rho} = \sum_{n=0}^{\infty} p_n |e^{(n)}\rangle\langle e^{(n)}| \; ; \; \sum_{n=0}^{\infty} p_n = 1 \; ; \; 0 \leq p_i \leq 1 \;\; \forall i$$

It clearly represents a set of states $\{|e^{(n)}\rangle\}$ with probability distribution $\{p_n\}$. Thus, the idea of using mixed states as a basis replaces the "fixed" vectors usually used with "noisy vectors" or "random vectors"

- We first consider the thermodynamic density operator for a canonical ensemble of single-fequency (ω) harmonic oscillators with Hamiltonian $\hat{H}_0 = \hbar\omega(\hat{a}_0^\dagger \hat{a}_0 + 1/2)$ in thermal equilibrium at temperature $T \equiv \frac{1}{k_B \beta}$

$$\hat{\rho}_0(\beta) \equiv \frac{e^{-\beta \hat{H}_0}}{\mathrm{Tr}\left(e^{-\beta \hat{H}_0}\right)} = \left(1 - e^{-\beta\hbar\omega}\right) e^{-\beta\hbar\omega \hat{a}_0^\dagger \hat{a}_0}$$

Of particular importance to us is the displaced linear harmonic oscillator with Hamiltonian

$$\hat{H}_z \equiv \hat{D}(z) \hat{H}_0 \hat{D}^\dagger(z) = \hbar\omega\left[(\hat{a}_0^\dagger - z^* \mathbb{1})(\hat{a}_0 - z\mathbb{1}) + 1/2\right]$$
$$\equiv \hbar\omega\left(\hat{a}_z^\dagger \hat{a}_z + 1/2\right)$$

Its density operator is

$$\hat{\rho}(z;\beta) \equiv \frac{e^{-\beta \hat{H}_z}}{\mathrm{Tr}\left(e^{-\beta \hat{H}_z}\right)} = \left(1-e^{-\beta\hbar\omega}\right) e^{-\beta\hbar\omega \hat{a}^{\dagger}_z \hat{a}_z}$$

or

$$\boxed{\begin{aligned} \hat{\rho}(z;\beta) &= \sum_{n=0}^{\infty} P_n(\beta)\, |n;z\rangle\langle n;z| \\ |n;z\rangle &\equiv \hat{D}(z)|n\rangle \\ P_n(\beta) &= \left(1-e^{-\beta\hbar\omega}\right) e^{-n\beta\hbar\omega} \end{aligned}}$$

Boltzmann distrib^{n}: for single-frequency (ω) oscillator

We readily check its zero-temperature limit

$$\lim_{\beta\to\infty} \hat{\rho}(z;\beta) = |z\rangle\langle z| \equiv \hat{\Pi}(z),$$

the projector onto coherent states $|z\rangle$

- We can easily compute the P- and Q-representations of $\hat{\rho}(\zeta;\beta)$:

$$\boxed{\begin{aligned} Q(\hat{\rho}(\zeta;\beta);z) &= \left(1-e^{-\beta\hbar\omega}\right) \exp\left[-\left(1-e^{-\beta\hbar\omega}\right)|z-\zeta|^2\right] \\ P(\hat{\rho}(\zeta;\beta);z) &= \left(e^{\beta\hbar\omega}-1\right) \exp\left[-\left(e^{\beta\hbar\omega}-1\right)|z-\zeta|^2\right] \end{aligned}}$$

$$\Downarrow$$

$$\int \frac{d^2\zeta}{\pi}\, Q(\hat{\rho}(\zeta;\beta);z) = 1 = Q(\mathbb{1};z)$$

$$\int \frac{d^2\zeta}{\pi}\, P(\hat{\rho}(\zeta;\beta);z) = 1 = P(\mathbb{1},z)$$

$$\Downarrow$$

$$\int \frac{d^2\zeta}{\pi}\, \hat{\rho}(\zeta;\beta) = \mathbb{1}$$

- Thus, we've proved the all-imporant resolution of the identity $\boxed{\int \frac{d^2 z}{\pi}\, \hat{\rho}(z;\beta) = \mathbb{1}}$

which enables us to expand arbitrary states $|f\rangle$ as:

$$\boxed{\begin{gathered}|f\rangle = \sum_{n=0}^{\infty} P_n \int \frac{d^2z}{\pi} f(n;z)\,|n;z\rangle \\ f(n;z) \equiv \langle n;z|f\rangle\end{gathered}}$$

"noisy basis" $\{|n;z\rangle\}$

- Relation to Glauber-Lachs formalism in quantum optics

- Glauber showed that the P-representation for a field which is a superposition of 2 separate fields with individual P-representatives $P_1(z)$ and $P_2(z)$ is given by their convolution:

$$P(z) \equiv \int \frac{d^2z'}{\pi} P_1(z') P_2(z-z')$$

- In the case of a single ω-mode, the P-representation of an oscillator of corresponding frequency in the coherent state $|\zeta\rangle$ is clearly

$$P_1(z) = \pi\, \delta^{(2)}(z-\zeta)$$

- Furthermore, Glauber has also shown that the P-representative of the corresponding mode of a random incoherent field with average number of photons $\bar{n}$ is:

$$P_2(z) = \frac{1}{\bar{n}} e^{-|z|^2/\bar{n}}$$

→ Hence, the P-representative for the mixed incoherent and coherent field is

$$P(z) = \frac{1}{\bar{n}} e^{-|z-\zeta|^2/\bar{n}}$$

which provides the basis for the Glauber-Lachs formalism in quantum optics

- In our case we have essentially, for a single-frequency (ω) mode, a superposition of a coherent field in a state $|\zeta\rangle$ with a corresponding thermal (incoherent) black-body radiation field at temperature T, for which the mean number of thermal photons, n_T, is:

$$\bar{n} \to n_T \equiv \langle \hat{n}_0 \rangle = \frac{\mathrm{Tr}\left[\hat{a}_0^{+}\hat{a}_0\, e^{-\beta\hbar\omega\hat{a}_0^{+}\hat{a}_0}\right]}{\mathrm{Tr}\left[e^{-\beta\hbar\omega\hat{a}_0^{+}\hat{a}_0}\right]}$$

$$= -\frac{d}{dx}\ln\left[\mathrm{Tr}\, e^{-x\hat{a}_0^{+}\hat{a}_0}\right] \; ; \; x \equiv \beta\hbar\omega$$

$$= -\frac{d}{dx}\ln(1-e^{-x})$$

$$\Rightarrow \bar{n} \to n_T = \frac{1}{e^{\beta\hbar\omega}-1}$$

the usual Planck distribution

(and hence in complete agreement with the above expression for $P(\hat{\rho}(\zeta;\beta);z)$ on p.7)

—∥—

• Generalized P- and Q-representations for nonzero temperatures

- As we've seen, the formalism of P- and Q-representations is based on pure coherent states. Although they form an overcomplete basis they are practically usable because of the existence of a resolution of the identity operator in terms of them.
- Thus, since our new mixed coherent states $\{\hat{\rho}(z;\beta)\}$ generalize the pure coherent states to nonzero temperatures, $T \neq 0$, we are led to enquire whether we can generalize the P- and Q- representatives of arbitrary operators $\hat{\Theta}$ to the $T\neq 0$ case, in terms of them.
- Specifically, we define

$$\boxed{Q(\hat{\Theta};z;\beta) \equiv \mathrm{Tr}\left[\hat{\rho}(z;\beta)\hat{\Theta}\right]}$$

and we hypothesize the expansion for an arbitrary (trace-class) operator $\hat{\Theta}$:

$$\boxed{\hat{\Theta} = \int \frac{d^2z}{\pi} P(\hat{\Theta};z;\beta)\hat{\rho}(z;\beta)}$$

Clearly: $\lim\limits_{\beta\to\infty} Q(\hat{\Theta};z;\beta) = Q(\hat{\Theta};z)$

$\lim\limits_{\beta\to\infty} P(\hat{\Theta};z;\beta) = P(\hat{\Theta};z)$

- The above 2 relations give immediately:

$$Q(\hat{\Theta};z;\beta) = \int \frac{d^2z'}{\pi} \mathrm{Tr}\left[\hat{\rho}(z;\beta)\hat{\rho}(z';\beta')\right] P(\hat{\Theta};z';\beta')$$

as a relation between the generalized P- and Q-representatives of an arbitrary operator $\hat{\Theta}$

We find:

$$\mathrm{Tr}\left[\hat{\rho}(z;\beta)\hat{\rho}(z';\beta')\right] = \frac{2}{g(\beta,\beta')}\exp\left[-\frac{2|z-z'|^2}{g(\beta,\beta')}\right]$$

$$g(\beta,\beta') \equiv \frac{\sinh\left[\frac{1}{2}(\beta+\beta')\hbar\omega\right]}{\sinh(\frac{1}{2}\beta\hbar\omega)\sinh(\frac{1}{2}\beta'\hbar\omega)}$$

$\Rightarrow$

$$Q(\hat{\Theta};z;\beta) = \exp\left[\tfrac{1}{8}g(\beta,\beta')\Delta_z\right]P(\hat{\Theta};z;\beta')$$

$$\Leftrightarrow \tilde{Q}(\hat{\Theta};\varpi;\beta) = \exp\left[-\tfrac{1}{8}g(\beta,\beta')|\varpi|^2\right]\tilde{P}(\hat{\Theta};\varpi;\beta')$$

<u>Special case</u>: $\beta' \to +\infty$

$$\tilde{Q}(\hat{\Theta};\varpi;\beta) = \exp\left[-\tfrac{1}{4}(e^{\beta\hbar\omega}-1)^{-1}|\varpi|^2\right]\tilde{Q}(\hat{\Theta};\varpi)$$

from which we find:

$$\tilde{Q}(\hat{\Theta};\varpi;\beta) = \exp\left[-\tfrac{1}{8}g(\beta,-\beta')|\varpi|^2\right]\tilde{Q}(\hat{\Theta};\varpi;\beta')$$

$$\tilde{P}(\hat{\Theta};\varpi;\beta) = \exp\left[+\tfrac{1}{8}g(\beta,-\beta')|\varpi|^2\right]\tilde{P}(\hat{\Theta};\varpi;\beta')$$

Finally, we also easily find the very important relations

$$\boxed{\begin{aligned}\tilde{Q}(\hat{\Theta};\varpi;\beta) &= \exp\left[-\tfrac{1}{8}\coth(\tfrac{1}{2}\beta\hbar\omega)|\varpi|^2\right]\tilde{W}(\hat{\Theta};\varpi)\\ \tilde{P}(\hat{\Theta};\varpi;\beta) &= \exp\left[+\tfrac{1}{8}\coth(\tfrac{1}{2}\beta\hbar\omega)|\varpi|^2\right]\tilde{W}(\hat{\Theta};\varpi)\end{aligned}}$$

$$\Leftrightarrow \quad \boxed{\begin{aligned}Q(\hat{\Theta};z;\beta) &= \exp\left[+\tfrac{1}{8}\coth(\tfrac{1}{2}\beta\hbar\omega)\Delta_z\right]W(\hat{\Theta};z)\\ P(\Theta;z;\beta) &= \exp\left[-\tfrac{1}{8}\coth(\tfrac{1}{2}\beta\hbar\omega)\Delta_z\right]W(\hat{\Theta};z)\end{aligned}}$$

$\Rightarrow$ The generalized ($T \neq 0$) P-representation may be viewed as the analytic continuation to negative temperatures of the generalized Q-representation, and vice versa!

→ Fascinating (and very suggestive)!

Two-boson (squeezed) SU(1,1) pure coherent states

We now wish to generalize our thermal mixed states to a broader class, if possible. To this end first consider a separate class of pure coherent states built on the SU(1,1) group. The SU(1,1) Lie algebra is:

$$\{\hat{K}_x,\hat{K}_y,\hat{K}_z\} : [\hat{K}_x,\hat{K}_y] = -i\hat{K}_z \; ; \; [\hat{K}_y,\hat{K}_z] = +i\hat{K}_x \; ; \; [\hat{K}_z,\hat{K}_x] = +i\hat{K}_y$$

or, in terms of $K_\pm \equiv K_x \pm i K_y \; ; \; \hat{K}_0 \equiv \hat{K}_z$

$$\{\hat{K}_+,\hat{K}_-,\hat{K}_0\} : \; [\hat{K}_0,\hat{K}_\pm] = \pm\hat{K}_\pm \; ; \; [\hat{K}_+,\hat{K}_-] = -2\hat{K}_0$$

[Note: just like SU(2) apart from the all-important minus signs!]

Casimir operator: $\hat{K}^2 \equiv \hat{K}_z^2 - \hat{K}_x^2 - \hat{K}_y^2 = \hat{K}_0^2 - \frac{1}{2}(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+)$

One can readily check that the standard basis is is $|k;\mu\rangle$; $\mu = \pm(k+n)$; $n = 0, 1, 2, \cdots$

(similar to SU(2) but here $|\mu| \geq k$)

$$\hat{K}_0 |k;\mu\rangle = \mu |k;\mu\rangle$$

$$\hat{K}_+ |k;\mu\rangle = \sqrt{(n+1)(n+2k)}\, |k;\mu+1\rangle$$

$$\hat{K}_- |k;\mu\rangle = \sqrt{n(n+2k-1)}\, |k;\mu-1\rangle$$

$$\hat{K}^2 |k;\mu\rangle = k(k-1)\, |k;\mu\rangle$$

(the group, unlike SU(2), is non-compact; and there is no restriction in general on the values of $k \in \mathbb{R}$)

- The so-called discrete series of representations takes the values $k = \frac{1}{2}, 1, \frac{3}{2}, \cdots$
- The group generators are now the so-called squeezing operators: $\hat{S}(\xi,\lambda) \equiv e^{(\xi\hat{K}_+ - \xi^*\hat{K}_-)}\underbrace{e^{i\lambda\hat{K}_0}}$

usually ignore, since it just gives a phase factor

There are various ways of writing $\hat{S}$, using various (easily proven) SU(1,1) disentangling relations:

$$\boxed{\begin{aligned}\hat{S}(z) \equiv \hat{S}(t,\phi) &\equiv e^{i\phi\hat{K}_z} e^{it\hat{K}_y} e^{-i\phi\hat{K}_z} \\ &= e^{\frac{t}{2}(e^{i\phi}\hat{K}_+ - e^{-i\phi}\hat{K}_-)} \\ &= e^{it(\sin\phi.\hat{K}_x + \cos\phi.\hat{K}_y)} \equiv e^{it\hat{n}\cdot\hat{K}} \\ &= e^{z\hat{K}_+}(1-|z|^2)^{\hat{K}_0} e^{-z^*\hat{K}_-} \\ \text{where } z \equiv \tanh\left(\tfrac{t}{2}\right)e^{i\phi} &\;;\; \hat{n} = (\sin\phi, \cos\phi, 0)\end{aligned}}$$

- so far, this is completely abstract (general)
- but, now let's examine 3 different realizations of the SU(1,1) algebra

- Realization (1)

A common realization of the SU(1,1) algebra arises in the context of single-mode (2-photon) squeezed states:

$$\hat{K}_+ \rightarrow \tfrac{1}{2}(\hat{a}_0^\dagger)^2 \;;\; \hat{K}_- \rightarrow \tfrac{1}{2}\hat{a}_0^2 \;;\; \hat{K}_0 \rightarrow \tfrac{1}{2}(\hat{a}_0^\dagger\hat{a}_0 + \tfrac{1}{2})$$

$$\Rightarrow \hat{K}^2 = -\tfrac{3}{16}\mathbb{1} \Rightarrow k(k-1) = -\tfrac{3}{16}$$

$\Rightarrow$ this is a $k = (1/4, 3/4)$ representation of SU(1,1)

The mode of action of $\hat{S}(z)$ is now

$$\hat{S}(z)\,\hat{a}_0\,\hat{S}^\dagger(z) = \frac{1}{\sqrt{1-|z|^2}}(\hat{a}_0 - z\hat{a}_0^\dagger) \equiv \hat{b}_z$$

$$\hat{S}(z)\,\hat{a}_0^\dagger\,\hat{S}^\dagger(z) = \frac{1}{\sqrt{1-|z|^2}}(\hat{a}_0^\dagger - z^*\hat{a}_0) \equiv b_z^\dagger$$

— homogeneous Bogoliubov $X^{\underline{ns}}$.

- Realization (2)

The SU(1,1) algebra can also be realized in the context of a harmonic oscillator $(\hat{a}_0 ; \hat{a}_0^{\dagger})$ as follows:

(Holstein-Primakoff)
$$\hat{K}_+ \to \hat{a}_0^{\dagger}(\hat{n}_0 + 2k\mathbb{1})^{1/2}$$
$$\hat{K}_- \to (\hat{n}_0 + 2k\mathbb{1})^{1/2}\hat{a}_0 \quad ; \hat{n}_0 \equiv \hat{a}_0^{\dagger}\hat{a}_0$$
$$\hat{K}_0 \to \hat{n}_0 + k\mathbb{1}$$

(— it's easy to check that these operators acting on the set of eigenstates $\{|n\rangle\}$ of $\hat{n}_0 \equiv \hat{a}_0^{\dagger}\hat{a}_0$, reproduce the proper SU(1,1) relations given above)

— We also readily check that in the $\hat{k}$-realization of SU(1,1), if we let the group operators $\hat{S}_k(t,\phi) \equiv \hat{S}_k(z)$ act on the vacuum state $|0\rangle$ we generate a new class of SU(1,1) (Perelomov) coherent states

$$\boxed{\begin{aligned} |z;k\rangle &\equiv \hat{S}_k(z)|0\rangle \\ &= (1-|z|^2)^k \sum_{n=0}^{\infty} [p(k,n)]^{1/2} z^n |n\rangle \; ; |z|<1 \\ \text{where } \; p(k,n) &\equiv \frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)} \end{aligned}}$$

— We note that $\overset{p_n(k,|z|^2)\equiv}{p_n} \equiv |\langle n|z;k\rangle|^2 = \frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)}|z|^{2n}(1-|z|^2)^{2k}$

is just the negative binomial distribution

$$\left(\text{i.e., } (1-|z|^2)^{-2k} = \sum_{n=0}^{\infty} \frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)} |z|^{2n} \right)$$

— Note: The case $k = 1/2$ is just our previous thermal (Planck) distribution with $|z|^2 = e^{-\beta\hbar\omega}$

— Very importantly, these states are also coherent states because they give a resolution of the identity operator:

$$\frac{2k-1}{\pi}\int_D d\mu(z)\,|z;k\rangle\langle z;k| = \mathbb{1} \quad ; \; k > 1/2$$

where $D \equiv$ unit disk $(|z|<1)$

$$d\mu(z) = \frac{d^2z}{(1-|z|^2)^2}$$ Lobachevsky metric

i.e., the invariant measure on the unit disk $|z|<1$

(NB: The case $k = 1/2$ handled as a limiting case $k = 1/2 + |\epsilon|$)

- as expected in association with the hyperbolic (Lobachevsky) geometry of SU(1,1) with area $\frac{d^2z}{(1-|z|^2)^2}$; distance $\frac{dz}{(1-|z|^2)}$

  (i.e., large distances as $|z| \to 1^-$; edge of disk)

(– Möbius conformal mappings $w = \frac{az+b}{b^*z+a^*}$; $|a|^2-|b|^2=1$ (3 parameters)

map D onto D : group is $SU(1,1)|Z_2$)

- overlap of 2 coherent states

$$\langle z;k|z';k\rangle = \frac{(1-|z|^2)^k(1-|z'|^2)^k}{(1-z^*z')^{2k}}$$

$$\Rightarrow |\langle z;k|z';k\rangle|^2 = \left[1 + \frac{|z-z'|^2}{(1-|z|^2)(1-|z'|^2)}\right]^{-2k}$$

hyperbolic distance

[c.f.: Glauber coherent states: $|\langle z'|z\rangle|^2 = e^{-|z-z'|^2}$]

flatworld distance

[NB: The Glauber coherent states are the limiting case $k \to \infty$; $z \to 0$; kz = const. of these SU(1,1) Perelomov coherent states]

- the analogue of the Bargmann holomorphic representation is now the k-analytic representation in the unit disk:

$$|f\rangle = \sum_{n=0}^{\infty} f_n|n\rangle \; ; \; \sum_{n=0}^{\infty} |f_n|^2 = 1$$

is represented by $|f\rangle \to f(z;k) = \sum_{n=0}^{\infty} f_n\,[p(k,n)]^{1/2} z^n$; $|z|<1$

$$= (1-|z|^2)^{-k}\langle z^*;k|f\rangle$$

– the Hardy space $H_2(D)$

- using the resolution of the identity, we find

$$\langle f^* | g \rangle = \frac{2k-1}{\pi} \int_D d^2 z \left(1-|z|^2\right)^{2k-2} f(z;k)\, g(z^*;k)$$

and the analogous reproducing kernel relation is

$$\frac{2k-1}{\pi} \int_D d^2 z' \left(1-|z'|^2\right)^{2k-2} \left(1-z'^* z\right)^{-2k} f(z';k) = f(z;k)$$

which one can easily prove to be valid for all functions $f(z;k) \in H_2(D)$, i.e., analytic in the unit disk and square-integrable in above sense

- Realization (3)

We now consider a two-mode harmonic oscillator Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, and its subspaces spanned by the joint number eigenstates

$$\mathcal{H}_\ell \equiv \left\{ |N+\ell, N\rangle \,;\; N = \max(0,-\ell), \cdots, \infty \right\}$$

where ℓ and N are integers, and

$$|N+\ell, N\rangle \equiv \underset{\text{type A oscillator}}{|N+\ell\rangle_A} \, \underset{\text{type B oscillator}}{|N\rangle_B}$$

Clearly : $\mathcal{H}_A \otimes \mathcal{H}_B = \bigoplus_{\ell=-\infty}^{\infty} \mathcal{H}_\ell$

Let operators $(\hat{a}_1^\dagger, a_1)$ act in $\mathcal{H}_A$; $(\hat{a}_2^\dagger, \hat{a}_2)$ act in $\mathcal{H}_B$

We readily check that
$$\begin{cases} \hat{K}_+ \rightarrow \hat{a}_1^\dagger \hat{a}_2^\dagger \\ K_- \rightarrow \hat{a}_1 \hat{a}_2 \\ \hat{K}_0 \rightarrow \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1) \end{cases}$$

$$\Rightarrow \hat{K}^2 \equiv \hat{K}_0^2 - \frac{1}{2}(\hat{K}_+ \hat{K}_- + \hat{K}_- \hat{K}_+) \rightarrow \frac{1}{4}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)^2 - \frac{1}{4}$$

provides a 2-mode realization of SU(1,1)

In the Hilbert space $\mathcal{H}_\ell$, the mutual eigenfunctions of $\hat{K}_0, \hat{K}^2$ are the states $|N+\ell, N\rangle$

$$K_0|N+\ell,N\rangle = [N+\tfrac{1}{2}(\ell+1)]\,|N+\ell,N\rangle$$

$$K^2|N+\ell,N\rangle = \tfrac{1}{4}(\ell^2-1)\,|N+\ell,N\rangle$$

Comparison with the general relations $\hat{K}_0|k;\mu\rangle = k|k;\mu\rangle$
$\hat{K}^2|k;\mu\rangle = k(k-1)|k;\mu\rangle$

shows that the above operators acting in $\mathcal{H}_\ell$ give the $k = \frac{1}{2}, 1, \frac{3}{2}, \cdots$ discrete series representations of $SU(1,1)$ with $k = \frac{1}{2}(1+|\ell|)$

$\Rightarrow$ the direct sum in $\mathcal{H}_A \otimes \mathcal{H}_B = \bigoplus_{\ell=-\infty}^{\infty} \mathcal{H}_\ell$

contains one $k=\frac{1}{2}$ representation, and two representations of each of $k = 1, \frac{3}{2}, 2, \cdots$

- Consider the projectors $\hat{\Pi}_\ell$ onto $\mathcal{H}_\ell$

$$\hat{\Pi}_\ell \equiv \sum_{N=\max(0,-\ell)}^{\infty} |N+\ell,N\rangle\langle N+\ell,N|$$

$$\Rightarrow \quad \hat{\Pi}_\ell \hat{\Pi}_m = \hat{\Pi}_\ell \delta_{\ell m}$$

$$\sum_{\ell=-\infty}^{\infty} \hat{\Pi}_\ell = \mathbb{1}$$

They are easily seen to commute with the above $SU(1,1)$ group operators $\{\hat{K}_+, \hat{K}_-, \hat{K}_0\}$, and hence with the corresponding two-mode squeezing operators $\hat{S}(t,\phi) \equiv \hat{S}(z)$: $[\hat{S}(z);\hat{\Pi}_\ell] = 0, \forall \ell$

$\Rightarrow$ $\hat{S}(t,\phi) = \hat{S}(z)$ leaves $\mathcal{H}_\ell$ invariant

- The mode of action of these squeezing operators on the fundamental creation and destruction operators is:

$$\hat{S}(z)\hat{a}_1\hat{S}^\dagger(z) = \frac{1}{\sqrt{1-|z|^2}}\left(\hat{a}_1 - z\hat{a}_2^\dagger\right) \equiv b_z^{(1)}$$

$$\hat{S}(z)\hat{a}_1^\dagger S^\dagger(z) = \frac{1}{\sqrt{1-|z|^2}}\left(\hat{a}_1^\dagger - z^* a_2\right) \equiv b_z^{(1)\dagger}$$

$$\hat{S}(z)\hat{a}_2\hat{S}^\dagger(z) = \frac{1}{\sqrt{1-|z|^2}}\left(\hat{a}_2 - z\hat{a}_1^\dagger\right) \equiv b_z^{(2)}$$

$$\hat{S}(z)\hat{a}_2^\dagger\hat{S}^\dagger(z) = \frac{1}{\sqrt{1-|z|^2}}\left(\hat{a}_2^\dagger - z^*\hat{a}_1\right) \equiv b_z^{(2)\dagger}$$

- Again, if we let this $\hat{S}(z)$ act on the lowest state $|\max(0,-\ell), \max(0,-\ell)+\ell\rangle$ in $\mathcal{H}_\ell$, we generate the corresponding 2-mode SU(1,1) coherent states in $\mathcal{H}_\ell$. For ease of discussion, let's restrict ourselves to $\ell \geq 0$ (– the case $\ell < 0$ is easily obtained by $A \rightleftarrows B$)

$\Rightarrow$

$$|\ell,0;z\rangle \equiv \hat{S}(z)|\ell,0\rangle$$
$$= \left(1-|z|^2\right)^{\frac{1}{2}(1+\ell)} \sum_{N=0}^{\infty} \left[\frac{(N+\ell)!}{N!\,\ell!}\right]^{1/2} z^N |N+\ell, N\rangle$$

$\ell > 0$

Let's now return to realization (2) of the pure SU(1,1) Perelomov coherent states, which generalise our Glauber coherent states, and see if we can correspondingly extend them to a new class of mixed coherent states, which (if they exist) will generalize our earlier thermal mixed coherent states.

Displaced negative-binomial mixed coherent states

Define:

$$\boxed{\begin{aligned} \hat\rho(k,r) &\equiv (1-r)^{2k}\sum_{n=0}^{\infty} p(k,n)\, r^n |n\rangle\langle n| \;;\; 0<r<1 \\ p(k,n) &\equiv \frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)} \;;\; k=\tfrac{1}{2},1,\tfrac{3}{2},\dots \\ &= \frac{(n+2k-1)!}{n!\,(2k-1)!} \end{aligned}}$$

- our previous thermal ~~[illegible]~~ mixed states are just the special case $k=\frac{1}{2}$

$$\hat\rho_{th}(r) \equiv \hat\rho(k=\tfrac{1}{2},r) = (1-r)\sum_{n=0}^{\infty} r^n |n\rangle\langle n|$$

$$= \left(1-e^{-\beta\hbar\omega}\right) e^{-\beta\hbar\omega \hat a_0^\dagger \hat a_0}$$

$$\beta\hbar\omega \equiv -\ln r$$

- interestingly, they may be expanded as

$$\hat\rho(k,r) = \int_0^{2\pi} \frac{d\theta}{2\pi} |z;k\rangle\langle z;k| \;\;;\;\; z=\sqrt{r}\,e^{i\theta}$$

- the average number of bosons ("photons") in this mixed state is $\bar n \equiv \mathrm{Tr}\left[\hat\rho(k,r)\hat a_0^\dagger \hat a_0\right] = \frac{2kr}{1-r}$
- $\hat\rho(k,r)$ can also be expressed in terms of $\hat\rho_{th}(r)$:

$$\hat\rho(k,r) = \frac{1}{\Gamma(2k)}\left(\frac{1}{r}-1\right)^{2k-1} \hat a_0^{2k-1}\, \hat\rho_{th}(r) \left(\hat a_0^\dagger\right)^{2k-1}$$

- special case: $\hat\rho(k,r=0) = |0\rangle\langle 0|$
- finally, as before, now displace $\hat\rho(k,r)$ to give the displaced negative-binomial mixed (coherent) states:

$$\boxed{\begin{aligned} \hat{\rho}(z;k,r) &\equiv \hat{D}(z)\,\hat{\rho}(k,r)\,\hat{D}^{\dagger}(z)\,; \quad \hat{D}(z) \equiv e^{z\hat{a}_0^{\dagger} - z^{*}\hat{a}_0} \\ &= (1-r)^{2k} \sum_{n=0}^{\infty} p(k,n)\, r^{n}\, |n;z\rangle\langle n;z| \;; \quad 0<r<1 \\ \text{where} \quad & |n;z\rangle \equiv \hat{D}(z)|n\rangle \end{aligned}}$$

- the average number of bosons in this mixed coherent state is

$$\bar{n} \equiv \mathrm{Tr}\left[\hat{\rho}(z;k,r)\,\hat{a}_0^{\dagger}\hat{a}_0\right] = \underbrace{|z|^2}_{\text{from coherent source}} + \underbrace{\frac{2kr}{1-r}}_{\text{from incoherent ('negative-binomial') noise}}$$

- once again, these states can also be regarded as mixed coherent states, because they give an easily proven resolution of the identity operator:

$$\boxed{\int \frac{d^2 z}{\pi}\, \hat{\rho}(z;k,r) = \mathbb{1}}$$

  - and hence everything we did for the thermal ($k=1/2$) case can be repeated here;
  - in particular, we can give further (k,r)-generalizations of P- and Q-representations

$$Q(\hat{\Theta};z;k,r) \equiv \mathrm{Tr}\left[\hat{\rho}(z;k,r)\,\Theta\right]\,;$$

$$\hat{\Theta} = \int \frac{d^2 z}{\pi}\, P(\hat{\Theta};z;k,r)\, \hat{\rho}(z;k,r)$$

    - with the case $r=0$ giving the ordinary P- and Q-representations; and
    - with $k=1/2$ giving our previous thermal results

  - Typical of the many interesting relations we find are:

$$\tilde{P}(\hat{\Theta};w;k,r) = \exp\left[\tfrac{1}{8}\left(\tfrac{1+r}{1-r}\right)|w|^2\right]\left[L_{2k-1}\left(\tfrac{|w|^2 r}{4(1-r)}\right)\right]^{-1}\tilde{W}(\hat{\Theta};w)$$

$$\tilde{Q}(\hat{\Theta};w;k,r) = \exp\left[-\tfrac{1}{8}\left(\tfrac{1+r}{1-r}\right)|w|^2\right] L_{2k-1}\left(\tfrac{|w|^2 r}{4(1-r)}\right)\tilde{W}(\hat{\Theta};w)$$

↑ Laguerre polynomial

and

$$\tilde{Q}(\hat{\Theta};w;k,r_1) = \exp\left[-\frac{|w|^2}{4}\frac{(1-r_1r_2)}{(1-r_1)(1-r_2)}\right] L_{2k-1}\left(\frac{|w|^2 r_1}{4(1-r_1)}\right) L_{2k-1}\left(\frac{|w|^2 r_2}{4(1-r_2)}\right)$$
$$\times\, \tilde{P}(\hat{\Theta};w;k,r_2)$$

* * *

- Finally, we also note the existence of another resolution of the identity in the variable r, for fixed z, k

$$(2k-1)\int_0^1 \frac{dr}{(1-r)^2}\,\hat{\rho}(z;k,r) = \mathbb{1} \quad ; \; k > \tfrac{1}{2}$$

(again with $k = \frac{1}{2}$ as a limit $k = \frac{1}{2} + |\epsilon|$)

and it might be fruitful to contemplate exploiting this relation also.

Generalization of thermofield dynamics

- Let's finally return to realization (3) of SU(1,1) and the corresponding two-mode pure SU(1,1) coherent states considered above

$$|\ell,0;z\rangle = \hat{S}(z)|\ell,0\rangle$$
$$= \left(1-|z|^2\right)^{\frac{1}{2}(1+\ell)} \sum_{n=0}^{\infty} \left[\frac{(n+\ell)!}{n!\,\ell!}\right]^{1/2} z^n\, |n+\ell,n\rangle$$

and the corresponding pure-state density operator in $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$\hat{\rho}(\ell,0;z) \equiv |\ell,0;z\rangle\langle \ell,0;z|$$

- Now, let's take the partial trace of this $\hat{\rho}$ with respect to either one of the 2 modes A or B:

$\ell \geq 0$

$$\boxed{\begin{aligned}\hat{\rho}_B(\ell,0;z) &\equiv \mathrm{Tr}_A\, \hat{\rho}(\ell,0;z)\\ &= \hat{\rho}\left(k=\tfrac{1+\ell}{2};\ r=|z|^2\right)\end{aligned}}$$

– just the previous negative-binomial mixed state

Similarly

$\ell \geq 0$

$$\boxed{\begin{aligned}\hat{\rho}_A(\ell,0;z) &\equiv \mathrm{Tr}_B\, \hat{\rho}(\ell,0;z)\\ &= \left(1-|z|^2\right)^{1+\ell} \sum_{n=0}^{\infty} \frac{(n+\ell)!}{n!\,\ell!}\, |z|^{2n}\, |n+\ell\rangle\langle n+\ell|\end{aligned}}$$

– a "shifted negative-binomial mixed state
(where "shifted" ≡ the series starts with the term $\propto |\ell\rangle\langle\ell|\,;\ \ell \geq 0$ and not $|0\rangle\langle 0|$

Only in the special case $\ell = 0$ (i.e, $k=1/2$) do both partial traces agree and give the thermal density operator

$$\hat{\rho}_A(0,0;z) = \hat{\rho}_B(0,0;z) = \hat{\rho}\left(k=\tfrac{1}{2},\ r=|z|^2\right) \equiv \hat{\rho}_{th}\left(r=|z|^2\right)$$

- Now, let $\hat{\Theta}_B$ be any operator in $\mathcal{H}_B$ and $\mathbb{1}_A$ the unit operator in $\mathcal{H}_A$ $\Rightarrow$

$$\boxed{\begin{aligned}\langle l,0;z|\mathbb{1}_A\otimes\hat{\Theta}_B|l,0;z\rangle &= \mathrm{Tr}\left[(\mathbb{1}_A\otimes\Theta)\hat{\rho}(l,0;z)\right]\\ &= \mathrm{Tr}_B\left[\hat{\Theta}_B\hat{\rho}_B(l,0;z)\right]\\ &= \mathrm{Tr}\left[\hat{\Theta}\hat{\rho}(k=\tfrac{1+l}{2};\tau=|z|^2)\right]\end{aligned}}$$

Special case $k=\frac{1}{2}$ $(l=0)$ gives just the basic relation of thermofield dynamics

$$\boxed{\begin{aligned}\langle 00;z|\mathbb{1}_A\otimes\hat{\Theta}_B|00;z\rangle &= \mathrm{Tr}\left[\hat{\Theta}\hat{\rho}_{th}(\tau=|z|^2)\right]\\ \tau=|z|^2 &= \exp(-\beta\hbar\omega)\end{aligned}}$$

where: the mode A is the so-called fictitious mode; and the SU(1,1) Perelomov coherent state $|0,0;z\rangle$ $(\in\mathcal{H}_0)$ is the so-called thermofield vacuum

→ essentially all of the results of thermofield dynamics now follow from this last relation and our two-mode squeezing formalism

→ we have now generalized thermofield dynamics (which occurs in $\mathcal{H}_0$; $k=\frac{1}{2}$) to a much more general negative-binomial field dynamics (in $\mathcal{H}_l$; $k=\frac{1}{2}(1+|l|)$

→ also of use in many other cases where the second ("fictitious") mode is physically very real

- an example is the "accelerated observer" formalism (see, e.g., R.F. Bishop and A. Vourdas, J. Phys. A 19 (1986), 2525) where the 2 modes correspond to observers moving in opposite directions and each of whom only has a partial knowledge of the other due to observing only a partial region of space-time
  ⇒ a very simple derivation of Hawking radiation from a black hole

→ in many other examples we also only have partial knowledge about the parameters, and the "real" and "fictitious" modes represent simply the known and unknown parameters, respectively. Here the Trace is taken with respect to the unknown modes and ⇒
the pure state in the big Hilbert space ($\mathcal{H}_A \otimes \mathcal{H}_B$) becomes a mixed state in the small Hilbert space ($\mathcal{H}_A$)

⇒ in all such cases there will be a need for a basis of mixed states in the small Hilbert space

⇔ such admixtures of coherence (pure states) with general noise or decoherence (thermal or more general) is often essentially unavoidable (either intrinsic or due to practical limitations)