A Symmetry Principle for Topological Quantum Order

Gerardo Ortiz
Department of Physics - Indiana University

Towards an understanding of a new paradigm in the physics of matter

Zohar Nussinov
Washington University - St. Louis

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Orders in Matter:Invariance Principles

Understanding the properties of the Phases of Matter by using Symmetry Principles allows us to characterize them in terms of Universal behaviors

- Global Symmetry Breaking Orders (e.g. Magnets)

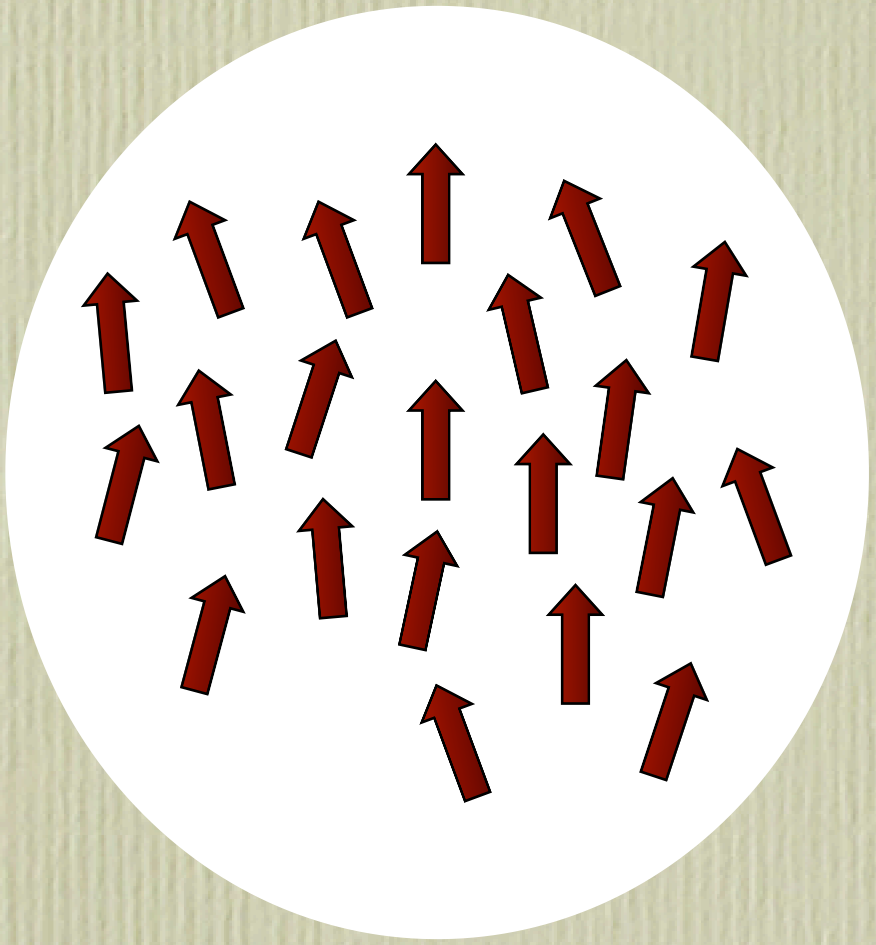
  Landau paradigm to matter classification
  in terms of an Order Parameter

- The new paradigm of Topological Order (e.g. Quantum Hall, Gauge Theories, Spin Liquids, String-Net models) - no obvious broken symmetry

  What characterizes these new orders ?
  Non-local Order Parameters?????

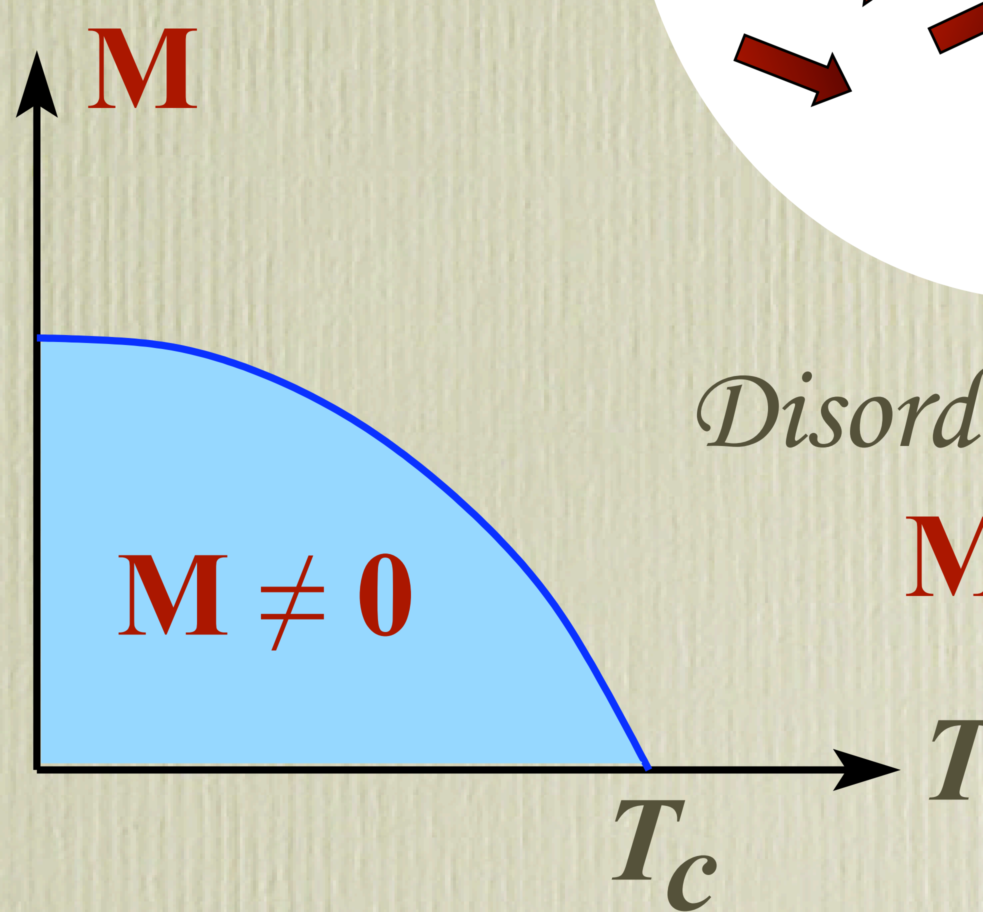
Symmetry and Phase Transitions

$T < T_c$

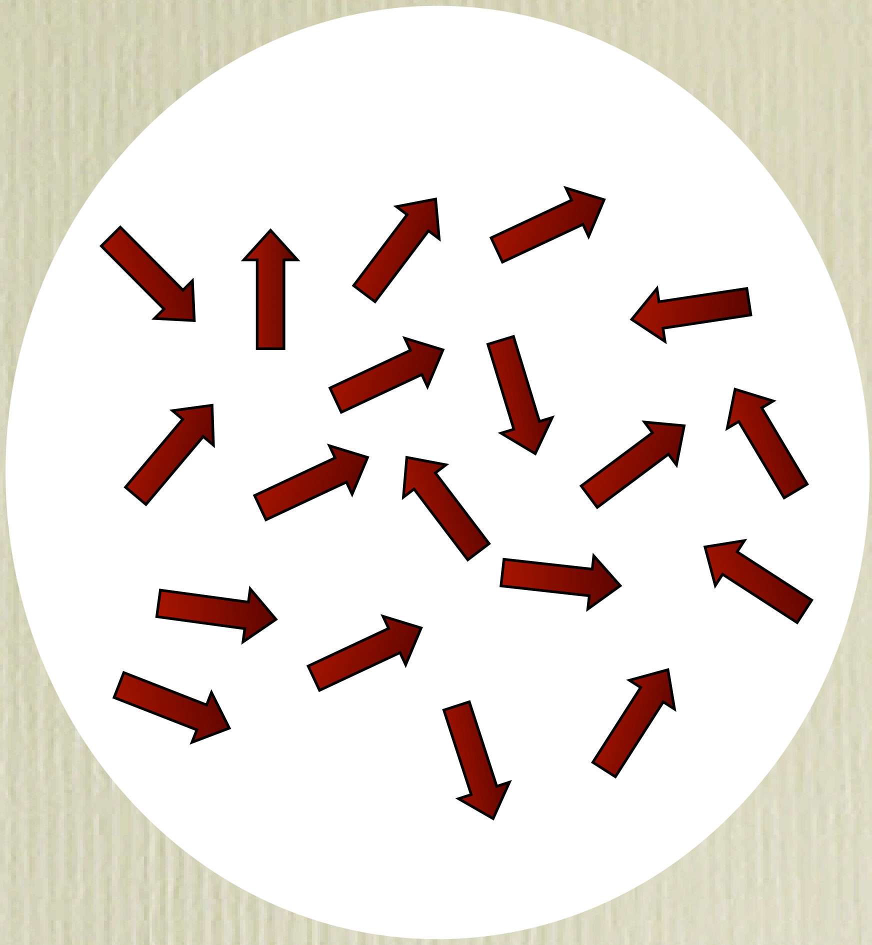


Broken Symmetry Phase

$\mathbf{M \neq 0}$



$T > T_c$



Disordered Phase

$\mathbf{M = 0}$

Local order parameters

In a ferromagnet, a local expectation value is different for different orthogonal ground states (GSs)

$$\langle g_\alpha | \hat{M} | g_\alpha \rangle \neq \langle g_\beta | \hat{M} | g_\beta \rangle \qquad T = 0$$

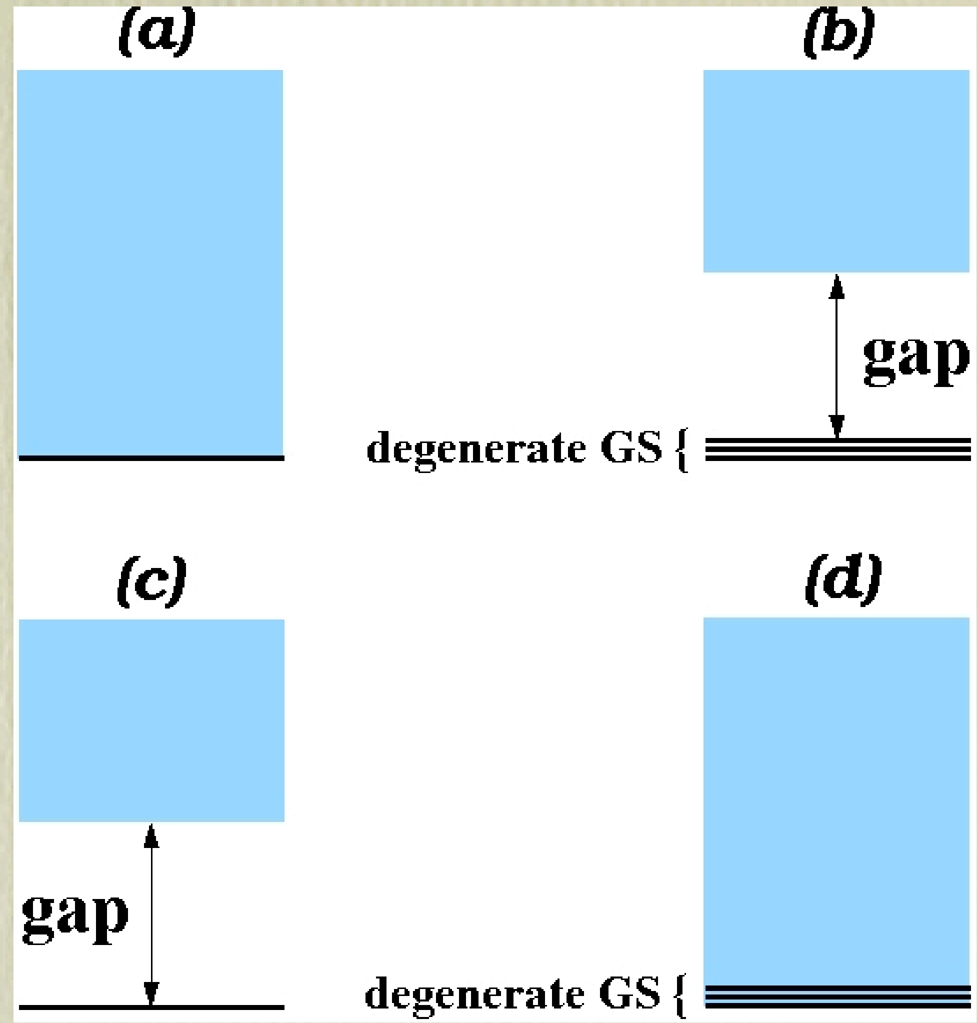
Applying different boundary conditions can lead, at sufficiently low temperatures to spontaneous symmetry breaking

$$\langle \hat{M} \rangle_\alpha \neq \langle \hat{M} \rangle_\beta \qquad T \neq 0$$

Local Measurements can distinguish the GSs

Why TQO?

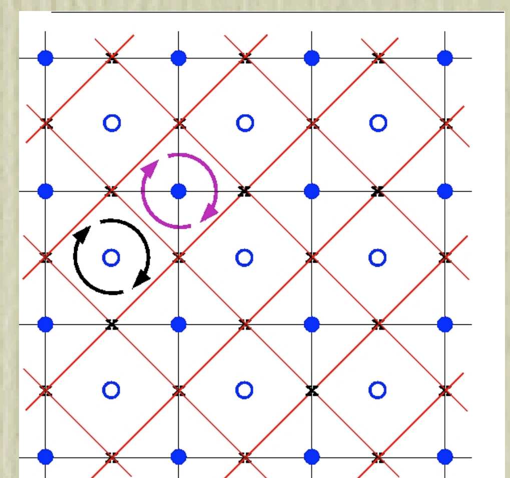
- New states of matter where the traditional Landau paradigm fails
- Topological Quantum Computation: Hardware Fault-tolerance
- Quantum simulator: Develop tools to simulate and detect these new states



Old Examples

- Fractional Quantum Hall Liquids

- Kitaev's Toric code model



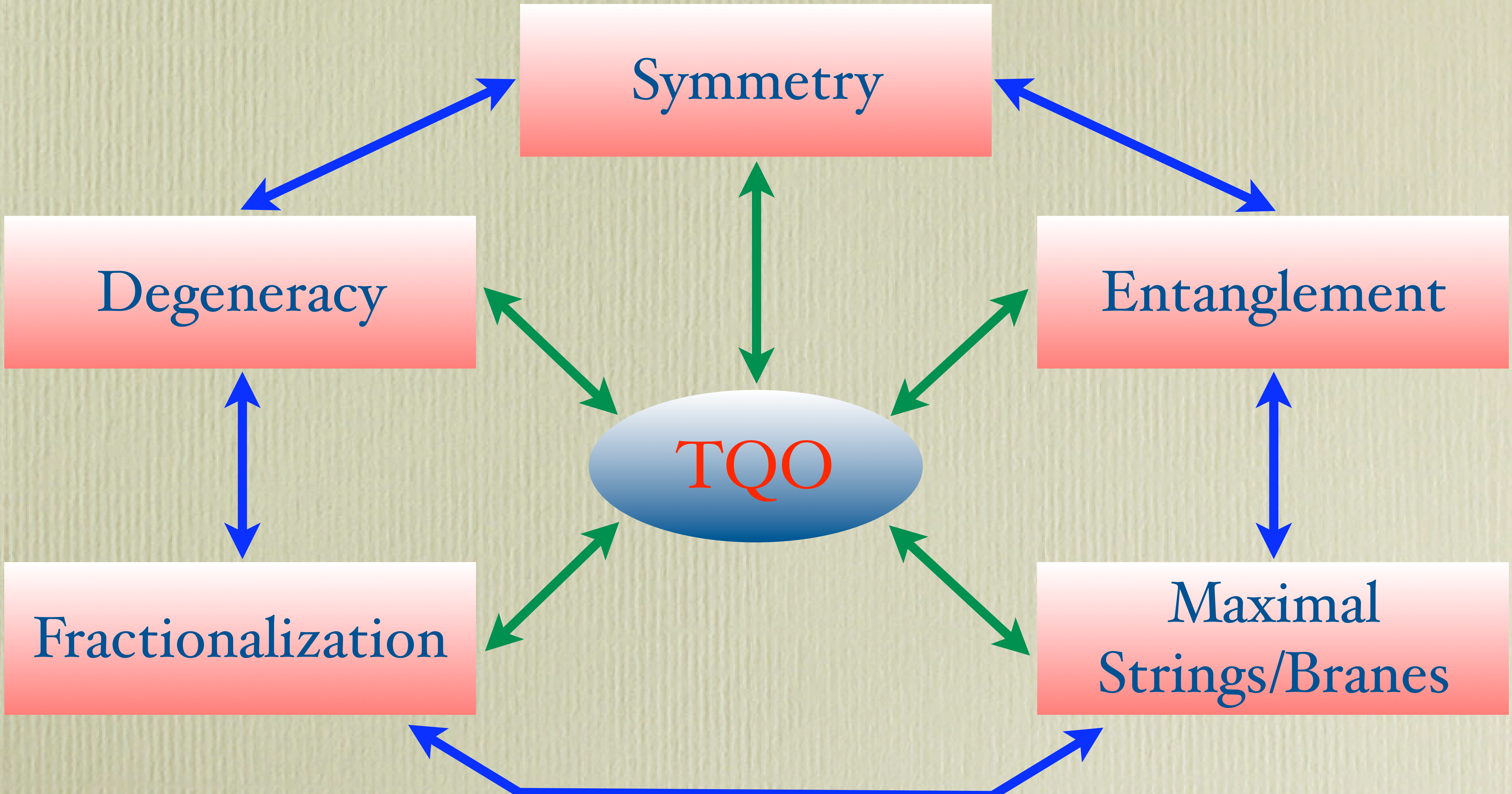
$$H = -\sum_s A_s - \sum_p B_p$$

$$A_s = \prod_{j \in \mathrm{star}(s)} \sigma_j^x \qquad B_p = \prod_{j \in \mathrm{boundary}(p)} \sigma_j^z$$

- $\mathbb{Z}_2$ Lattice Gauge Theory $\quad H = -\sum_p B_p$

- Some spin liquids

Concepts involved in TQO



What is TQO?

Colloquially TQO is often very loosely referred to as order whose

Order is evident only in non-local (topological) quantities

Our working definition: Robustness

Non-Distinguishability: Given a quasi-local operator $\hat{V}^m$

$$\langle g_\alpha | \hat{V}^m | g_\beta \rangle = c\, \delta_{\alpha\beta},\ \forall\, \alpha, \beta \in \mathcal{S}_0,$$

Perturbation Theory:

$$\langle g_\alpha | \underbrace{\hat{V} \bar{G}_0 \hat{V} \ldots \bar{G}_0 \hat{V}}_{m \text{ factors } \hat{V}} | g_\beta \rangle = c\, \delta_{\alpha\beta},\ \forall\, \alpha, \beta \in \mathcal{S}_0$$

$$\bar{G}_0 = (\epsilon_0 - H_0)^{-1} \hat{P}_\perp$$

What is the unifying physical principle behind TQO ?

Gauge-Like-Symmetries

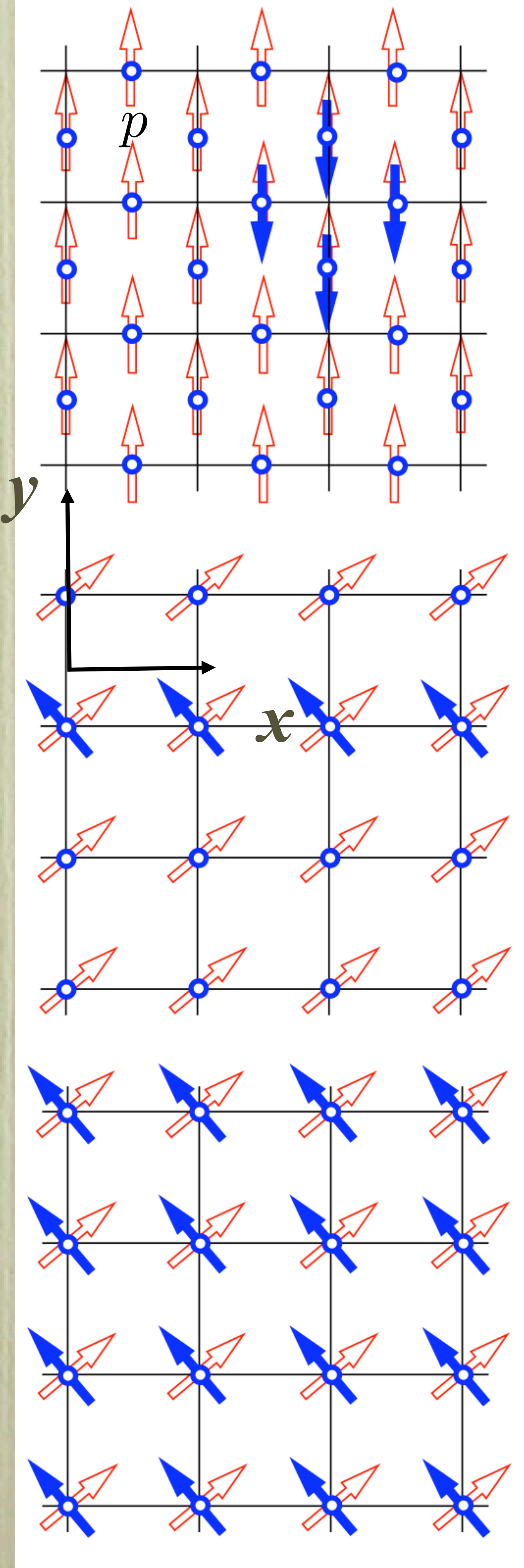
Given a *D*-dim theory:

A *d*-dim **GLS** is a group of transformations that leave the theory invariant such that the minimum non empty set of fields that are changed under the symmetry operation occupies a *d*-dim region

$$d \leq D$$

$d=0$ (Gauge)	$d < D$ (Gauge-Like)	$d=D$ (Global)

Group: $\mathcal{G}_d$



Gauge-Like-Symmetries $D = 2$

$d = 0$ (Ising Gauge Theory)

$$H = -K \sum_{p} \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li} \qquad G_i = \prod_{s \in \mathrm{nn}} \sigma^x_{is}$$

$d = 1$ (Orbital Compass Model)

$$H = -\sum_{i} [J_x \sigma^x_i \sigma^x_{i+\hat{e}_x} + J_y \sigma^y_i \sigma^y_{i+\hat{e}_y}]$$

$$O^x = \prod_{j \in C_x} i\sigma^x_j \qquad O^y = \prod_{j \in C_y} i\sigma^y_j$$

$d = D = 2$ (XY model)

$$H = -J \sum_{\langle ij \rangle} [\sigma^x_i \sigma^x_j + \sigma^y_i \sigma^y_j]$$

$$U(\theta) = \prod_j \exp[-(i/2)\theta\sigma^z_j]$$

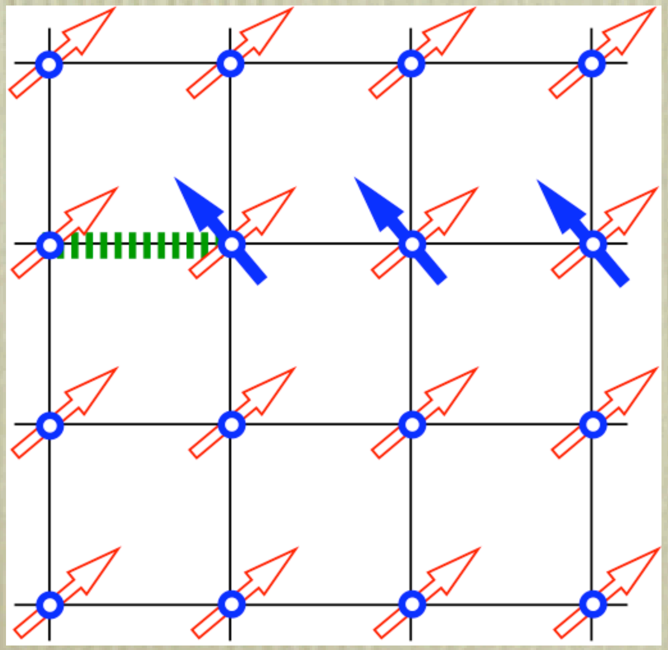
d-GLSs and Topological Phases

There is a connection between Topological Phases and the group generators of *d*-GLSs and its Topological defects

$d = 1$ (*D*=2 Orbital Compass Model) C_x : closed path

$$O^x = e^{i\frac{\pi}{2}\sum_{j\in C_x}\sigma_j^x} = \mathcal{P}e^{i\oint_{C_x}\vec{A}\cdot d\vec{s}}$$

Symmetries are linking operators: $O_\mu|g_\alpha\rangle = |g_\beta\rangle$



Topological defect: C_+ : open path

Defect-Antidefect pair creation

$$D^x = e^{i\frac{\pi}{2}\sum_{j\in C_+}\sigma_j^x} = \mathcal{P}e^{i\int_{C_+}\vec{A}\cdot d\vec{s}}$$

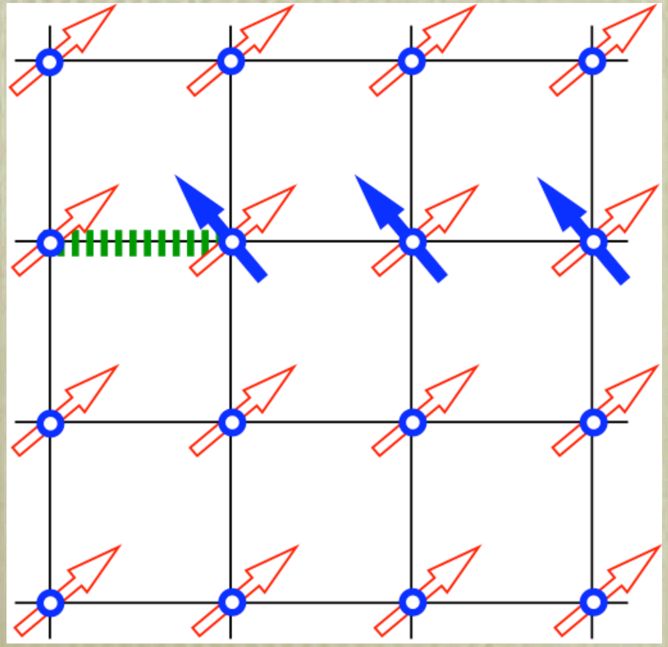
Physical Consequences

For a D-dim system, $d < D$ GLSs lead to
dimensional reduction

- Conservation Laws within d-dim regions:
  Additional conserved currents

- Topological terms that appear in $d+1$ also appear in $D+1$

- Freely propagating d-dim topological defects



$d=1$ soliton in the $D=2$ orbital compass model
(Finite Energy cost)

To Break or not to Break

Can we spontaneously break a *d*-GLS in a *D*-dim system ?

From the Generalized Elitzur's Theorem: (finite-range int.)

For non-$\mathcal{G}_d$-invariant quantities

- *d=0* SSB is forbidden
- *d=1* SSB is forbidden
- *d=2* (continuous) SSB is forbidden

  d=2 (discrete) SSB may be broken
- *d=2* (continuous with a gap) SSB is forbidden even at $T=0$

Transitions and crossovers are signaled by symmetry-invariant string/brane or Wilson-like loops

Fundamental Theorem Linking TQO and GLSs

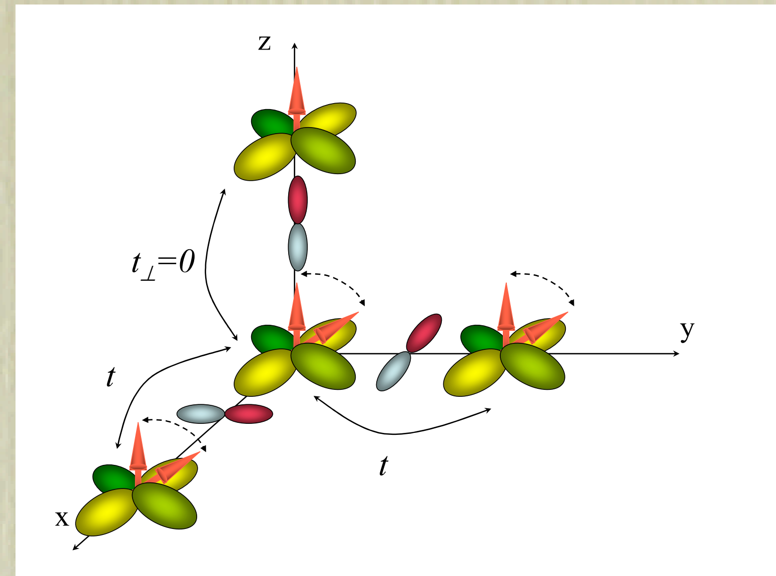
(Z. Nussinov and G. Ortiz, cond-mat/0605316, 0702377)

Any physical system which displays T=0 TQO, and interactions of finite range and strength, in which all GSs (satisfying the non-distinguishability condition) can be linked by discrete d < 2 or continuous d <3 GLSs, has TQO at all temperatures.

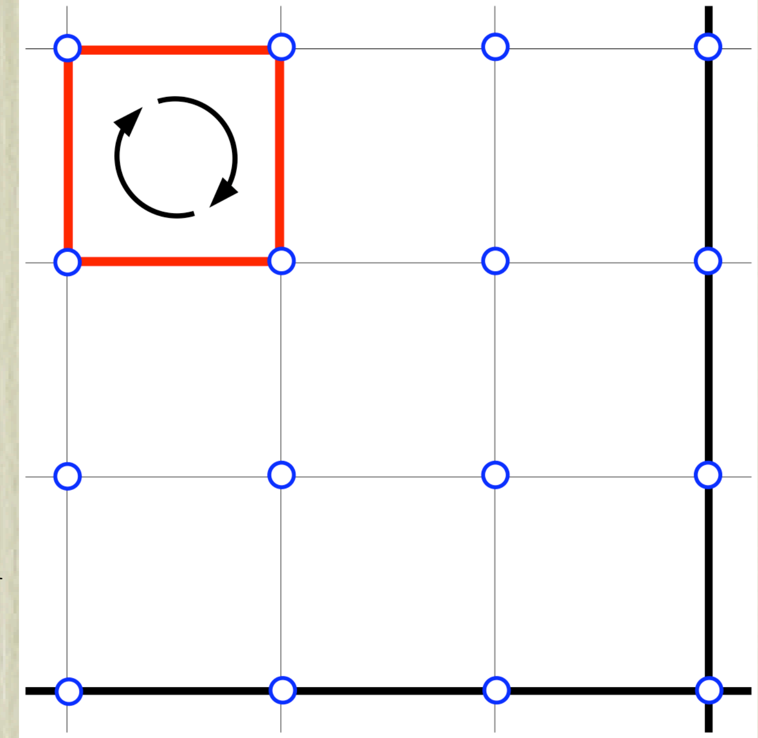
(d-GLSs with $d < D$ can mandate the absence of SSB)

New Examples

- Kugel-Khomskii Hamiltonians
  (Transition metals)

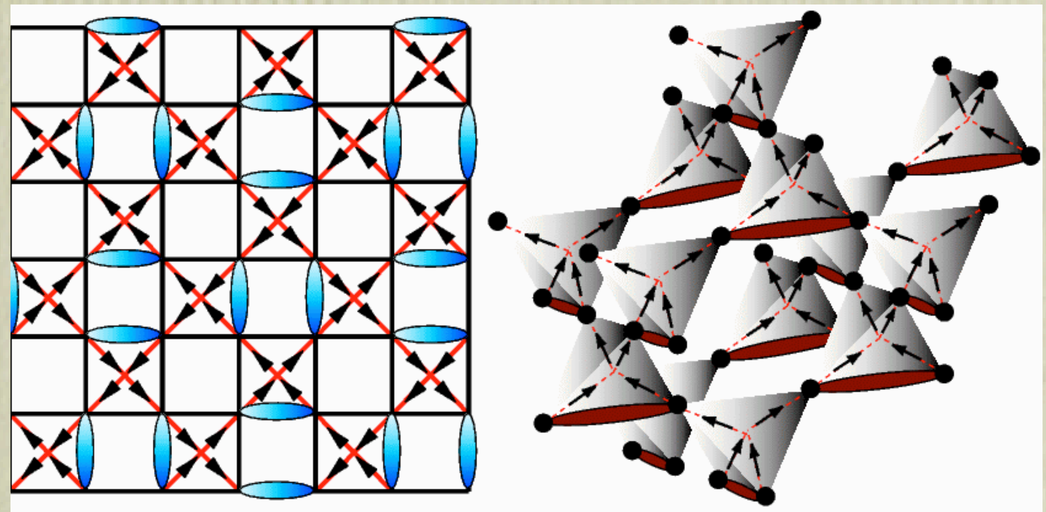


- Superconducting (p+ip) arrays
  (Sr_2RuO_4)



(p+ip) model $\leftrightarrow$ D=2 orbital compass model

- Klein spin models



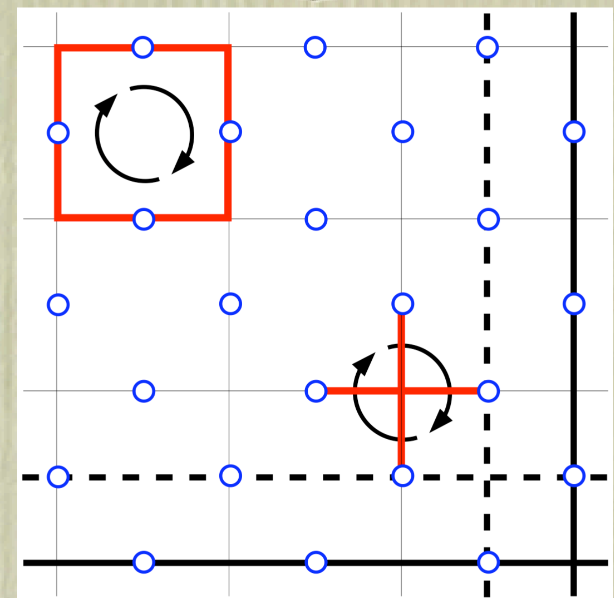
How do we mathematically characterize TQO ?

Insufficient criteria:

- Hamiltonian Spectrum: TQO is a property of states (Duality mappings disentangle the non-local order)
- Topological Entanglement Entropy (Deviation from an Area law)
- String/Brane Correlations: Long-range order of non-local operators

TQO is a property of States not of the Spectrum

Kitaev's toric code model:



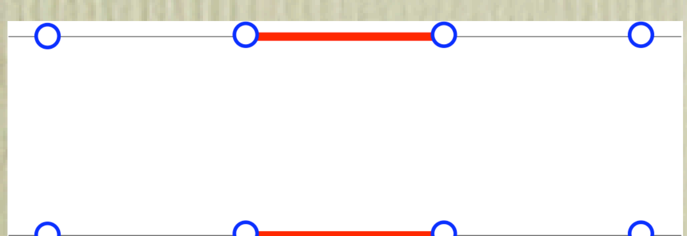
$$H_K = -\sum_s A_s - \sum_p B_p$$

$$A_s = \prod_{ij \in star(s)} \sigma^x_{ij}$$

$$B_p = \prod_{ij \in boundary(p)} \sigma^z_{ij}$$

Duality mappings: Non-local
(Identical spectra)

2 Ising chains:

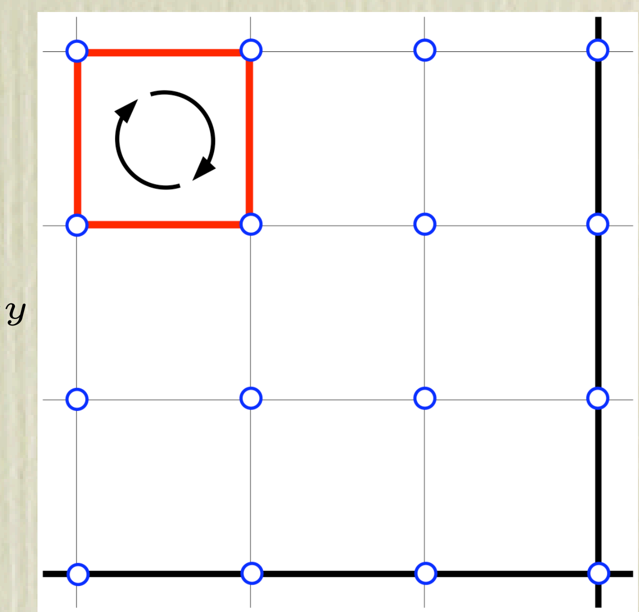


$$H_I = -\sum_s \sigma^z_s \sigma^z_{s+1} - \sum_p \sigma^z_p \sigma^z_{p+1}$$

(Nussinov-Ortiz 2006)

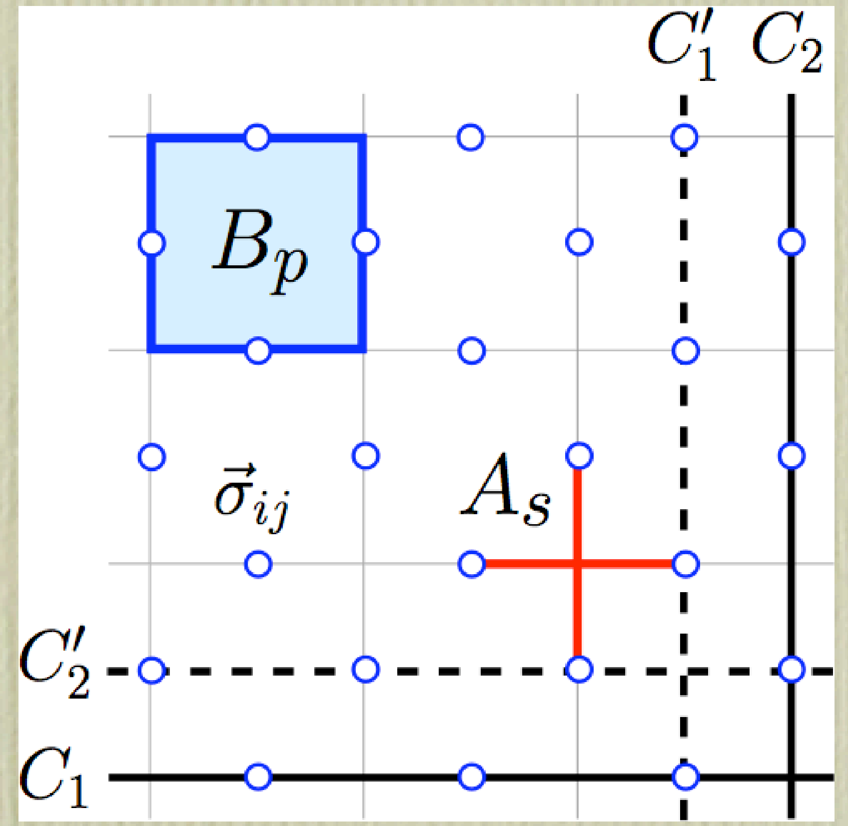
Wen's plaquette model:

$$H_W = -\sum_i \sigma^x_i \sigma^y_{i+\hat{e}_x} \sigma^x_{i+\hat{e}_x+\hat{e}_y} \sigma^y_{i+\hat{e}_y}$$



Thermal Fragility

In TQO systems, which have a gap,
does temperature preclude protection of information?



$$H = -\sum_s A_s - \sum_p B_p$$

$$A_s = \prod_{ij \in \text{star}(s)} \sigma^x_{ij} \qquad B_p = \prod_{ij \in \text{plaquette}(p)} \sigma^z_{ij}$$

$$X_{1,2} = \prod_{ij \in C'_{1,2}} \sigma^x_{ij} \qquad Z_{1,2} = \prod_{ij \in C_{1,2}} \sigma^z_{ij}$$

$$\{X_i, Z_i\} = 0 \; , \; [X_i, Z_j] = 0$$

Free-energy is analytic

Thermal Fragility

For a finite size: By Symmetry

$$\langle Z_1 \rangle = \langle Z_2 \rangle = \langle X_1 \rangle = \langle X_2 \rangle = 0$$

Partition function (2 Ising chains):

$$Z = \mathrm{tr}\left[\exp[-\beta(H - \sum_{i=1,2} (h_{x,i} X_i + h_{z,i} Z_i))] \right]$$

$$= [(2\cosh\beta)^{N_s} + (2\sinh\beta)^{N_s}]^2 \cosh\beta h_1 \cosh\beta h_2$$

$$h_i = \sqrt{h_{x,i}^2 + h_{z,i}^2}$$

$$\langle Z_i \rangle = \lim_{h_{z,i}\to 0^+} \frac{\partial}{\partial(\beta h_{z,i})} \ln Z = \lim_{h_{z,i}\to 0^+} \frac{h_{z,i}}{h_i} \tanh(\beta h_i)$$

$$= 0$$

$$\langle X_i \rangle = \lim_{h_{x,i}\to 0^+} \frac{\partial}{\partial(\beta h_{x,i})} \ln Z = \lim_{h_{x,i}\to 0^+} \frac{h_{x,i}}{h_i} \tanh(\beta h_i)$$

What have we done and proved?

Most significant results:

- Provide a unifying framework
- Fundamental Theorem: A sufficient symmetry condition to have TQO is that the system displays low *d*-dim GLSs
- *d*-dim ($d<D$) GLSs lead to dimensional reduction
- *d*-GLSs can enforce high dimensional fractionalization, unusual topological indices (& related Berry phases)

What have we done and proved?

- The devil is not in the spectrum
- Thermal effects seem to impose severe restrictions on several current suggestions for topological quantum computing (Thermal fragility)
- General entangled systems have string (or higher dimensional "brane") correlators which decay more slowly than the usual two-point correlators
- A goal is to use the symmetry principles to engineer new model Hamiltonians that can be easily realized experimentally.

What remains to be done?

Most significant questions:

- How do we characterize and classify TQO?
  (Entanglement entropy? Generalized entanglement?)
- How do we measure TQO? Experimental probes?
- Most importantly for quantum memories: Conditions under which TQO is protected from thermal effects?
- What are TQO states useful for?
  Quantum orders vs Functionalities